Chapter 4

The Small-Open-Economy Real-Business-Cycle Model

In the previous two chapters, we arrived at the conclusion that a model driven by productivity shocks can explain the observed countercyclicality of the trade balance. We also established that two features of the model are important for making this prediction possible. First, productivity shocks must be sufficiently persistent. Second, capital adjustment costs must not be too strong. In this chapter, we extend the model of the previous chapter by allowing for three features that make its structure more realistic: endogenous labor supply and demand, uncertainty in the technology shock process, and capital depreciation. The resulting theoretical framework is known as the Small-Open-Economy Real-Business-Cycle model, or, succinctly, the SOE-RBC model.

4.1 The Model

Consider a small open economy populated by an infinite number of identical households with preferences described by the utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, h_t), \tag{4.1}$$

where c_t denotes consumption, h_t denotes hours worked, $\beta \in (0,1)$ is the subjective discount factor, and U is a period utility function, which is assumed to be increasing in its first argument, decreasing in its second argument, and concave. The symbol E_t denotes the expectations operator conditional on information available in period t.

The period-by-period budget constraint of the representative household is given by

$$d_t = (1 + r_{t-1}) d_{t-1} - y_t + c_t + i_t + \Phi(k_{t+1} - k_t), \tag{4.2}$$

where d_t denotes the household's debt position at the end of period t, r_t denotes the interest rate at which domestic residents can borrow in period t, y_t denotes domestic output, i_t denotes gross investment, and k_t denotes physical capital. The function $\Phi(\cdot)$ is meant to capture capital adjustment costs and is assumed to satisfy $\Phi(0) = \Phi'(0) = 0$ and $\Phi''(0) > 0$. Small open economy models typically include capital adjustment costs to avoid excessive investment volatility in response to variations in the productivity of domestic capital or in the foreign interest rate. The restrictions imposed on Φ and Φ' ensure that in the steady state adjustment costs are nil and the relative price of capital goods in terms of consumption goods is unity. Note that here adjustment costs are expressed in terms of final goods. Alternatively, one could assume that adjustment costs take the form of lost capital goods (see exercise 4.9).

Output is produced by means of a linearly homogeneous production function that takes capital and labor services as inputs,

$$y_t = A_t F(k_t, h_t), \tag{4.3}$$

where A_t is an exogenous and stochastic productivity shock. This shock represents the single source of aggregate fluctuations in the present model. The stock of capital evolves according to

$$k_{t+1} = (1-\delta)k_t + i_t, \tag{4.4}$$

where $\delta \in (0,1)$ denotes the rate of depreciation of physical capital.

Households choose processes $\{c_t, h_t, y_t, i_t, k_{t+1}, d_t\}_{t=0}^{\infty}$ to maximize the utility function (4.1) subject to (4.2)-(4.4) and a no-Ponzi constraint of the form

$$\lim_{j\to\infty} E_t \frac{d_{t+j}}{\prod_{s=0}^{j}(1+r_s)} \leq 0. \tag{4.5}$$

Use equations (4.3) and (4.4) to eliminate, respectively, y_t and i_t from the sequential budget constraint (4.2). This operation yields

$$d_t = (1+r_{t-1})d_{t-1} - A_t F(k_t, h_t) + c_t + k_{t+1} - (1-\delta)k_t + \Phi(k_{t+1} - k_t). \tag{4.6}$$

The Lagrangian corresponding to the household's maximization problem is

$$\begin{aligned} \mathcal{L} \quad = \quad & E_0 \sum_{t=0}^{\infty} \beta^t \left\{ U(c_t, h_t) \right. \\ & \left. + \lambda_t \left[A_t F(k_t, h_t) + (1-\delta)k_t + d_t - c_t - (1+r_{t-1})d_{t-1} - k_{t+1} - \Phi(k_{t+1} - k_t) \right] \right\}, \end{aligned}$$

where $\beta^t \lambda_t$ denotes the Lagrange multiplier associated with the sequential budget constraint (4.6). The first-order conditions associated with the household's maximization problem are (4.5) holding with equality, (4.6), and

$$\lambda_t = \beta(1 + r_t)E_t\lambda_{t+1} \tag{4.7}$$

$$U_c(c_t, h_t) = \lambda_t \tag{4.8}$$

$$-U_h(c_t, h_t) = \lambda_t A_t F_h(k_t, h_t) \tag{4.9}$$

$$\lambda_t[1 + \Phi'(k_{t+1} - k_t)] = \beta E_t\lambda_{t+1}\left[A_{t+1}F_k(k_{t+1}, h_{t+1}) + 1 - \delta + \Phi'(k_{t+2} - k_{t+1})\right]. \tag{4.10}$$

Optimality conditions (4.7), (4.8), and (4.10) are familiar from chapter 3. Optimality condition (4.9) equates the supply of labor to the demand for labor. To put it in a more familiar form, divide (4.9) by (4.8) to eliminate λ_t. This yields

$$-\frac{U_h(c_t, h_t)}{U_c(c_t, h_t)} = A_t F_h(k_t, h_t). \tag{4.11}$$

The left-hand side of this expression is the household's labor supply schedule. It is the marginal rate of substitution between leisure and consumption, which is increasing in hours worked, holding the level of consumption constant.[1] The right-hand side of (4.11) is the marginal product of labor, which, in a decentralized version of this model equals the demand for labor. The marginal product of labor is decreasing in labor, holding constant the level of capital.

The law of motion of the productivity shock is assumed to be given by the first-order autoregressive process

$$\ln A_{t+1} = \rho \ln A_t + \widetilde{\eta}\epsilon_{t+1}, \tag{4.12}$$

where ϵ_t is assumed to be exogenous, stochastic, and i.i.d., with mean zero and unit standard deviation, the parameter $\widetilde{\eta}$ defines the standard deviation of the innovations to productivity, and

[1] A sufficient condition for $-U_h/U_c$ to be increasing in h_t holding c_t constant is $U_{ch} < 0$, and the necessary and sufficient condition is $U_{hh}/U_h > U_{ch}/U_c$.

the parameter $\rho \in (-1, 1)$ measures the serial correlation of the technology shock. According to this expression, the expected value of the productivity shock in period $t+1$ conditional on information available in period t is a fraction ρ of the current productivity shock,

$$E_t \ln A_{t+1} = \rho \ln A_t. \tag{4.13}$$

More generally, the assumed AR(1) structure of the productivity shock implies that the its expected value j periods ahead conditional on current information is a fraction ρ^j of its present value, $E_t \ln A_{t+j} = \rho^j A_t$. In other words, $\ln A_t$ is always expected to converge to zero at the rate ρ.

4.1.1 Inducing Stationarity: External Debt-Elastic Interest Rate (EDEIR)

In chapters 2 and 3 we saw that the equilibrium of a small open economy with one internationally traded bond and a constant interest rate satisfying $\beta(1+r) = 1$ features a random walk in consumption, net external debt, and the trade balance. Under perfect foresight, that model predicts that the steady state levels of debt, consumption, and the trade balance depend on initial conditions, such as the initial level of debt itself. This does not mean that the deterministic steady state is indeterminate. Rather, it means that the steady state is history dependent.

The nonstationarity of the small open economy model complicates the task of approximating equilibrium dynamics, because available approximation techniques require stationarity of the state variables. Here, we follow Schmitt-Grohé and Uribe (2003) and induce stationarity by making the interest rate debt elastic.[2]

Specifically, we assume that the interest rate faced by domestic agents, r_t, is increasing in the

[2] In section 4.10 we study various alternative ways to induce stationarity.

country's cross-sectional average of debt, which we denote by $\widetilde{d}_t$. Formally, r_t is given by

$$r_t = r^* + p(\widetilde{d}_t), \tag{4.14}$$

where r^* denotes the world interest rate and $p(\cdot)$ is a country-specific interest rate premium. Households take the evolution of $\widetilde{d}_t$ as exogenously given. For simplicity, we assume that the world interest rate, r^*, is constant. The function $p(\cdot)$ is assumed to be strictly increasing. As we will see shortly, the assumption of a debt-elastic interest rate premium gives rise to a steady state of the model that is independent of initial conditions. In addition, this assumption ensures that a first-order approximation of the equilibrium dynamics converge to the true equilibrium dynamics as the supports of the underlying shocks become small.

The intuition why a debt-elastic interest rate induces stationarity is simple. A growing level of debt causes the country premium to rise inducing households to increase savings, which curbs debt growth. Similarly, if the external debt falls below its steady state level, the country premium falls inducing households to increase consumption and reduce savings, which fosters debt growth.

Here, we have motivated a debt-elastic interest rate on purely technical grounds. However, this feature is also of interest for empirical and theoretical reasons. In chapters 5 and 6, we argue on econometric grounds that data from emerging countries favor a significantly debt-sensitive interest rate. From a theoretical point of view, a debt-elastic interest rate is of interest because it represents a simple way to capture the presence of financial frictions. In chapter 13, we provide micro-foundations to this interpretation in the context of models with imperfect enforcement of international debt contracts.

4.1.2 Equilibrium

Because agents are assumed to be identical, in equilibrium the cross-sectional average level of debt must be equal to the individual level of debt, that is,

$$\widetilde{d}_t = d_t. \tag{4.15}$$

Use equations (4.8), (4.14), and (4.15) to eliminate λ_t, r_t, and $\widetilde{d}_t$ from (4.5), (4.6), (4.7), and (4.10) to obtain

$$d_t = [1 + r^* + p(d_{t-1})]d_{t-1} + c_t + k_{t+1} - (1-\delta)k_t + \Phi(k_{t+1} - k_t) - A_t F(k_t, h_t). \tag{4.16}$$

$$U_c(c_t, h_t) = \beta(1 + r^* + p(d_t))E_t U_c(c_{t+1}, h_{t+1}) \tag{4.17}$$

$$U_c(c_t, h_t)[1 + \Phi'(k_{t+1} - k_t)] = \beta E_t U_c(c_{t+1}, h_{t+1}) \left[A_{t+1} F_k(k_{t+1}, h_{t+1}) + 1 - \delta + \Phi'(k_{t+2} - k_{t+1})\right]. \tag{4.18}$$

$$\lim_{j\to\infty} E_t \frac{d_{t+j}}{\prod_{s=0}^{j}(1 + r^* + p(d_s))} = 0. \tag{4.19}$$

A competitive equilibrium is a set of processes $\{d_t, c_t, h_t, k_{t+1}, A_t\}$ satisfying (4.11), (4.12), and (4.16)-(4.19), given A_0, d_{-1}, and k_0, and the process $\{\epsilon_t\}_{t=0}^{\infty}$.

Given the equilibrium processes of consumption, hours, capital, and debt, output is obtained from equation (4.3), investment from equation (4.4), and the interest rate from equation (4.14) evaluated at $\widetilde{d}_t = d_t$. One can then construct the equilibrium process of the trade balance from the definition

$$tb_t \equiv y_t - c_t - i_t - \Phi(k_{t+1} - k_t), \tag{4.20}$$

where tb_t denotes the trade balance in period t. Finally, the current account is given by the sum of the trade balance and net investment income, that is,

$$ca_t = tb_t - r_{t-1}d_{t-1}. \tag{4.21}$$

Alternatively, one could construct the equilibrium process of the current account by using the fact that the current account measures the change in net foreign assets, that is,

$$ca_t = d_{t-1} - d_t. \tag{4.22}$$

4.2 Decentralization

The economy presented thus far assumes that production, employment, and the use of capital are all carried out within the household. Here, we present an alternative formulation in which all of these activities are performed in the marketplace. This formulation is known as the decentralized economy. A key result of this section is that the equilibrium conditions of the decentralized economy are identical to those of the centralized one.

4.2.1 Households in the Decentralized Economy

We assume that each period the household supplies h_t hours to the labor market. We also assume that the household owns shares of a firm that produces physical capital and rents it to firms that produce final goods. Let w_t denote the real wage, π_t the profit generated by capital-producing firms, s_t the number of shares of the capital producing firm owned by the household and p_t^s the price of each share. The household takes w_t, π_t, and p_t^s as exogenously given. Its period-by-period budget constraint can then be written as

$$d_t = (1 + r_{t-1})d_{t-1} + c_t + p_t^s(s_t - s_{t-1}) - s_{t-1}\pi_t - w_t h_t \tag{4.23}$$

The household chooses processes $\{c_t, h_t, d_t, s_t\}_{t=0}^{\infty}$ to maximize the utility function (4.1) subject to (4.5) and (4.23), taking as given the processes $\{r_t, w_t, \pi_t, p_t^s\}_{t=0}^{\infty}$ and the initial conditions $(1+r_{-1})d_{-1}$ and s_{-1}. The first-order conditions associated with the household's problem are (4.5) holding with equality, (4.7), (4.8), (4.23),

$$-\frac{U_h(c_t, h_t)}{U_c(c_t, h_t)} = w_t, \tag{4.24}$$

and

$$\lambda_t p_t^s = \beta E_t \lambda_{t+1}[p_{t+1}^s + \pi_{t+1}]. \tag{4.25}$$

The variable p_t^s represents a stock market index such as the S&P 500. The above Euler equation can be integrated forward to obtain

$$p_t^s = E_t \sum_{j=1}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} \pi_{t+j}, \tag{4.26}$$

which states that the value of the stock market in period t equals the present discounted value of future expected profits.

4.2.2 Firms Producing Final Goods

Firms produce final goods with labor and capital and operate in perfectly competitive markets. The production technology is given by

$$y_t = A_t F(k_t, h_t).$$

Profits in period t are given by

$$A_t F(k_t, h_t) - w_t h_t - u_t k_t.$$

Each period $t \geq 0$ the firm hires workers and rents capital to maximize profits. The first-order conditions associated with the firm's profit maximization problem are

$$A_t F_h(k_t, h_t) = w_t \tag{4.27}$$

and

$$A_t F_k(k_t, h_t) = u_t. \tag{4.28}$$

Because the production function is assumed to be homogeneous of degree one, profits are zero at all times. To see this multiply (4.27) by h_t, (4.28) by k_t and sum the resulting expressions to obtain $A_t F_h(k_t, h_t)h_t + A_t F_k(k_t, h_t)k_t = w_t h_t + u_t k_t$. By the assumed linear homogeneity of the production function the left hand side of this expression is equal to $A_t F(k_t, h_t)$. It then follows that the total cost of production equals output, or, that profits equal zero.

4.2.3 Firms Producing Capital Goods

Firms producing capital invest i_t units of final goods each period and are subject to adjustment costs $\Phi(k_{t+1} - k_t)$ measured in units of final goods. Each period, these firms rent the stock of capital to firms producing final goods at the rental rate u_t per unit. Profits of firms producing capital goods are then given by

$$\pi_t = u_t k_t - i_t - \Phi(k_{t+1} - k_t) \tag{4.29}$$

The problem of the firm producing capital goods is to choose processes $\{\pi_t, i_t, k_{t+1}\}_{t=0}^{\infty}$ maximize the present discounted value of profits

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{\lambda_t}{\lambda_0} \pi_t$$

subject to the law of motion of the capital stock given in equation (4.4) and the definition of profits given in equation (4.29), taking as given the processes $\{u_t, \lambda_t\}_{t=0}^{\infty}$ and the initial condition k_0. Note that profits are discounted using the factor $\beta^t \lambda_t/\lambda_0$, which is the value assigned by households to contingent payments of goods in period t in terms of units of goods in period 0. This way of discounting makes sense because households own the firms producing capital. Note further that the objective function of the firm is identical to the right-hand side of optimality condition (4.26). This means that the objective of the firm producing capital can be interpreted as maximizing the value of the firm in the stock market.

Using equation (4.4) to eliminate i_t from equation (4.29) and the resulting expression to eliminate π_t from the firm's objective function yields

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{\lambda_t}{\lambda_0} \left[(u_t + 1 - \delta)k_t - k_{t+1} - \Phi(k_{t+1} - k_t)\right]$$

The optimality condition with respect to k_{t+1} is then given by

$$\lambda_t[1 + \Phi'(k_{t+1} - k_t)] = \beta E_t \lambda_{t+1} \left[u_{t+1} + 1 - \delta + \Phi'(k_{t+2} - k_{t+1})\right]. \tag{4.30}$$

4.2.4 The Decentralized Equilibrium

We can normalize the number of shares to be one per household at all times. Thus, we have

$$s_t = 1. \tag{4.31}$$

A competitive equilibrium in the decentralized economy is then a set of processes $\{d_t, \widetilde{d}_t, c_t, p_t^s, s_t, r_t, \pi_t, h_t, w_t, \lambda_t, y_t, u_t, k_{t+1}, i_t, A_t\}_{t=0}^{\infty}$, satisfying (4.3), (4.4), (4.7), (4.8), (4.12), (4.14), (4.15), (4.19), (4.23), (4.24), and (4.26)-(4.31), given A_0, d_{-1}, and k_0, and the process $\{\epsilon_t\}_{t=0}^{\infty}$.

It is straightforward to see that the equations included in this definition can be combined to produce all of the equations conforming the equilibrium in the centralized economy defined in section 4.1.2. It can also be readily established that if all of the conditions for an equilibrium in the centralized economy are satisfied, then one can residually construct processes for market prices, profits, and share holdings, namely processes $\{w_t, u_t, p_t^s, \pi_t, s_t\}_{t=0}^{\infty}$, so that all of the equilibrium conditions of the decentralized economy listed here are satisfied. This completes the proof that the equilibrium conditions of the centralized and decentralized economies are identical.

4.3 Functional Forms

We assume that the period utility function takes the form

$$U(c,h) = \frac{G(c,h)^{1-\sigma} - 1}{1-\sigma}, \quad \sigma > 0,$$

with

$$G(c,h) = c - \frac{h^{\omega}}{\omega}, \quad \omega > 1.$$

The form of the subutility index $G(c,h)$ is due to Greenwood, Hercowitz, and Huffman (1988) and is typically referred to as GHH preferences. It implies that the labor supply (the marginal rate of substitution between consumption and leisure) is independent of the level of consumption. Specifically, under GHH preferences, equilibrium condition (4.24) becomes

$$h_t^{\omega-1} = w_t. \tag{4.32}$$

This labor supply schedule has a wage elasticity of $1/(\omega - 1)$ and is independent of c_t. GHH preferences were popularized in the open economy business cycle literature by Mendoza (1991).

The period utility function $U(c, h)$ displays constant relative risk aversion (CRRA) over the subutility index $G(c, h)$. The parameter σ measures the degree of relative risk aversion, and its reciprocal, $1/\sigma$, measures the intertemporal elasticity of substitution.

We adopt a Cobb-Douglas specification for the production function,

$$F(k, h) = k^{\alpha} h^{1-\alpha},$$

with $\alpha \in (0, 1)$. This specification implies a unitary elasticity of substitution between capital and labor. That is, a one percent increase in the wage to rental ratio, w_t/u_t, induces firms to increase the capital-labor ratio by one percent. To see this divide equation (4.27) by equation (4.28) and use the Cobb-Douglas form for the production function to obtain

$$\left(\frac{1-\alpha}{\alpha}\right) \frac{k_t}{h_t} = \frac{w_t}{u_t},$$

which implies that in equilibrium the capital-labor ratio is proportional to the wage to rental ratio. The Cobb-Douglas specification of the production function is widely used in the business-cycle literature.

The capital adjustment cost function is assumed to be quadratic,

$$\Phi(x) = \frac{\phi}{2} x^2,$$

with $\phi > 0$. This specification implies that net investment, whether positive or negative, generates resource costs.

Finally, we follow Schmitt-Grohé and Uribe (2003) and assume that the country interest rate premium takes the form

$$p(d) = \psi_1 \left(e^{d-\overline{d}} - 1\right),$$

where $\psi_1 > 0$ and $\bar{d}$ are parameters. According to this expression the country premium is an increasing and convex function of net external debt.

4.4 Deterministic Steady State

Assume that the variance of the innovation to the productivity shock, $\widetilde{\eta}$, is nil. We refer to such an environment as a deterministic economy. We define a deterministic steady state as an equilibrium of the deterministic economy in which all endogenous variables are constant over time.

The characterization the deterministic steady state is of interest for two reasons. First, the steady state facilitates the calibration of the model. This is because, to a first approximation, the deterministic steady state coincides with the average position of the model economy. In turn, often several structural parameters of the model are o calibrated to match average characteristics of the model economy, such as labor shares, consumption shares, and trade-balance-to-output ratios to their empirical counterparts. Second, the deterministic steady state is often used as a convenient point around which the equilibrium conditions of the stochastic economy are approximated.

For any variable we denote its steady-state value by removing the time subscript. Evaluating equilibrium condition (4.17) at the steady state yields

$$1 = \beta \left[1 + r^* + \psi_1 \left(e^{d-\overline{d}} - 1\right)\right].$$

Assume that

$$\beta(1 + r^*) = 1.$$

In the context of the present model, this assumption is a normalization, and is not necessary to ensure stationarity. Combining the above two restrictions one obtains

$$d = \overline{d}.$$

The steady-state version of (4.18) implies that

$$1 = \beta \left[\alpha \left(\frac{k}{h} \right)^{\alpha - 1} + 1 - \delta \right].$$

This expression delivers the steady-state capital-labor ratio, which we denote by κ. Formally,

$$\kappa \equiv \frac{k}{h} = \left(\frac{\beta^{-1} - 1 + \delta}{\alpha} \right)^{1/(\alpha - 1)}.$$

Using this expression to eliminate the capital-labor ratio from equilibrium condition (4.11) evaluated at the steady state, one obtains the following expression for the steady-state level of hours

$$h = \left[(1 - \alpha) \kappa^{\alpha} \right]^{1/(\omega - 1)}.$$

Given the steady-state values of labor and the capital-labor ratio, the steady-state level of capital is simply given by

$$k = \kappa h.$$

Finally, the steady-state level of consumption can be obtained by evaluating equilibrium condition (4.16) at the steady state. This yields

$$c = -r^* \overline{d} + \kappa^{\alpha} h - \delta k.$$

This completes the characterization of the deterministic steady state of the present economy.

4.5 Calibration

An important intermediate step in computing the quantitative predictions of a business-cycle model is to assign values to its structural parameters. There are two main ways to accomplish this step. One is econometric estimation by methods such as the generalized method of moments (GMM), impulse response matching, maximum likelihood, or likelihood-based Bayesian methods. We will explain and apply several of these econometric techniques in later chapters. The second approach, which we study here, is calibration. Almost always, business-cycle studies employ a combination of calibration and econometric estimation.

In general, the calibration method assigns values to the parameters of the model in three different ways: (a) Using sources unrelated to the macro data the model aims to explain. (b) By matching first moments of the data that the model aims to explain. (c) By matching second moments of the data the model aims to explain.

To illustrate how calibration works, we adapt the calibration strategy adopted in Mendoza (1991) to the present model. His SOE-RBC model aims to explain the Canadian business cycle. The time unit in the model is meant to be one year. In the present model, there are 10 parameters that need to be calibrated: σ, δ, r^*, α, $\overline{d}$, ω, ϕ, ψ_1, ρ, and $\widetilde{\eta}$. We separate these parameters into the three calibration categories described above.

(a) Parameters Calibrated Using Sources Unrelated To The Data The Model Aims To Explain

The parameters that fall in this category are the intertemporal elasticity of substitution, σ, the depreciation rate, δ, and the world interest rate, r^*. Based on parameter values widely used in related business-cycle studies, Mendoza sets σ equal to 2, δ equal to 0.1, and r^* equal to 4 percent per year.

(b) Parameters Set To Match First Moments Of The Data The Model Aims To Explain

In this category are the capital elasticity of the production function, α, and the parameter $\overline{d}$ pertaining to the country interest-rate premium. The parameter α is set to match the average labor share in Canada of 0.68. In the present model, the labor share, given by the ratio of labor income to output, or $w_t h_t / y_t$, equals $1 - \alpha$ at all times. To see this, note that in equilibrium, w_t equals the marginal product of labor, which, under the assumed Cobb-Douglas production function is given by $(1 - \alpha) y_t / h_t$.

The parameter $\overline{d}$ is set to match the observed average trade-balance-to-output ratio in Canada of 2 percent. Combining the definition of the trade balance given in equation (4.20) with the resource constraint (4.16) implies that in the steady state

$$tb = r^* \overline{d}.$$

This condition states that in the deterministic steady state the country must generate a trade surplus sufficiently large to service its external debt. Dividing both sides by steady-state output and solving for $\overline{d}$ yields

$$\overline{d} = \frac{tb/y}{r^*} y,$$

At this point we know that $tb/y = 0.02$ and that $r^* = 0.04$, but y remains unknown. From the derivation of the steady state presented in section 4.4, one can deduce that

$$y = \left[(1 - \alpha) \kappa^{\alpha \omega} \right]^{\frac{1}{\omega - 1}},$$

where $\kappa = [(\alpha / (r^* + \delta)]^{1/(1-\alpha)}$. The only unknown parameter in the expression for y, and therefore $\overline{d}$, is ω. Next, we discuss how the calibration strategy assigns values to ω and the remaining unknown structural parameters.

(c) Parameters Set To Match Second Moments Of The Data The Model Aims To Explain

This category of parameters contains ω, which governs the wage elasticity of labor supply, ϕ, which defines the magnitude of capital adjustment costs, ψ_1 which determines the debt sensitivity of the interest rate, and ρ and $\widetilde{\eta}$ defining, respectively, the persistence and volatility of the technology shock. The calibration strategy for these parameters is to match the following five second moments of the Canadian data at business-cycle frequency: A standard deviation of hours of 2.02 percent, a standard deviation of investment of 9.82 percent, a standard deviation of the trade-balance-to-output ratio of 1.87 percentage points, a serial correlation of output of 0.62, and a standard deviation of output of 2.81 percent.[3] These are natural targets, as their theoretical counterparts are directly linked to the parameters to be calibrated. In practice, this last step of the calibration procedure goes as follows: (i) Guess values for the five parameters in category (c). This automatically determines a value for $\overline{d}$. (ii) Approximate the equilibrium dynamics of the model. (We will discuss how to accomplished this task shortly.) (iii) Calculate the implied five second moments to be matched in (c). (iv) If the match between actual and predicted second moments is judged satisfactory, the procedure has concluded. If not, try a new guess for the five parameters to be calibrated and return to (i). There is a natural way to update the parameter guess. For instance, if the volatility of output predicted by the model is too low, raise the volatility of the innovation to the technology shock, $\widetilde{\eta}$. Similarly, if the volatility of investment is too high, increase the value of ϕ. And so on. In general, there are no guarantees of the existence of a set of parameter values that will produce an exact match between the targeted empirical second moments and their theoretical counterparts.

[3] The standard deviations of hours, investment, and output are measured in percent because (the cyclical components of) hours, investment, and output are measured as percent deviations of these indicators from trend.

Table 4.1: Calibration of the EDEIR Small Open RBC Economy

Parameter	σ	δ	r^*	α	$\overline{d}$	ω	ϕ	ψ_1	ρ	$\widetilde{\eta}$
Value	2	0.1	0.04	0.32	0.7442	1.455	0.028	0.000742	0.42	0.0129

So some notion of distance and tolerance is in order. The parameter values that result from this calibration procedure are shown in table 4.1.

It is important to note that the calibration strategy presented here is just one of many possible ones. For instance, we could place δ in category (b) and add the average investment share as a first moment of the data to be matched. Similarly, we could take the parameter ω out of category (c) and place it instead in category (a). To assign a value to ω parameter we could then use existing micro-econometric estimates of the Frisch elasticity of labor supply. Finally, a calibration approach that has been used extensively, especially in the early days of the RBC literature, is to place ρ and $\widetilde{\eta}$ into category (a) instead of (c). Under this approach, one uses Solow residuals as a proxy for the productivity shock A_t. Then one estimates a univariate representation of the Solow residual to obtain values for ρ and $\widetilde{\eta}$.

4.6 Approximating Equilibrium Dynamics

The competitive equilibrium of the SOE-RBC model is described by a system of nonlinear stochastic difference equations. Closed-form solutions to this type of systems are typically unavailable. We therefore must resort to an approximate solution. There exist a number of techniques that have been devised to solve such dynamic systems. The one we study in this section is based on a linear approximation of the equilibrium conditions.

It is important to chose carefully the base of the linearization. It is oftenappropriate to linearize the system with respect to the logarithm of some variables. This is known as log-linearization, and is useful for variables whose empirical counterparts are expressed in log (or percent) deviations from trend. In the present SOE-RBC model, this is the case with y_t, c_t, h_t, k_t, and A_t. For other variables, it is more natural to perform the linearization with respect to their levels, not with respect to their logs. This is the case, for instance, with net interest rates, like r_t, or variables that can take negative values, such as tb_t, ca_t, and d_t, or ratios, like the investment-to-output ratio.

Before performing the linearization of the equilibrium conditions of the SOE-RBC model, we briefly explain how to linearize a function with respect to a mix of bases, the log for some variables and the level for others. As an illustration, consider the expression

$$s_t = E_t m(u_t, v_t, z_{t+1}).$$

We wish to linearize this expression with respect to the logs of s_t, u_t, and z_{t+1}, and with respect to the level of v_t. To this end, let $\widehat{s}_t \equiv \ln(s_t/s)$, $\widehat{u}_t \equiv \ln(u_t/u)$, and $\widehat{z}_{t+1} \equiv \ln(z_{t+1}/z)$ denote the log-deviations of s_t, u_t, and z_{t+1} with respect to their respective deterministic steady-state values, denoted s, u, and z, and let $\widehat{v}_t \equiv v_t - v$ denote the deviation of v_t from its steady-state value, denoted v. Then, we can write the above expression as

$$se^{\widehat{s}_t} = E_t m\left(ue^{\widehat{u}_t}, \widehat{v}_t + v, ze^{\widehat{z}_{t+1}}\right).$$

The linerization results from differentiating the above expression with respect to $\widehat{s}_t$, $\widehat{u}_t$, $\widehat{v}_t$, and $\widehat{z}_{t+1}$ around their respective deterministic steady-state values. Note that the deterministic steady-state values of all hatted variables is zero. In performing the differentiation, recall that the expectation operation is an inegral, and that the differentiation of an integral with respect to variables appearing in the integrand is the integral of the differentiated integrand. Then the desired linear approximation is given by

$$s\widehat{s}_t = m_u u\widehat{u}_t + m_v \widehat{v}_t + m_z z E_t \widehat{z}_{t+1},$$

where m_u, m_v, and m_z denote the partial derivatives of $m(\cdot,\cdot,\cdot)$ with respect to u_t, v_t, and z_{t+1}, respectively, evaluated at the steady state (u, v, z). With this background, we now turn to the linearization of the equilibrium conditions of the SOE-RBC model.

We linearize the system with respect to the logs of c_t, h_t, k_t, and A_t, and with respect to the level of d_t. Accordingly, let $\widehat{x}_t \equiv \ln(x_t/x)$, for $x_t = c_t, h_t, k_t, A_t$, and $\widehat{d}_t \equiv d_t - d$. Then, the linearized version of equilibrium conditions (4.11), (4.13), and (4.16)-(4.18) is

$$[\epsilon_{hh} - \epsilon_{ch}]\widehat{h}_t + [\epsilon_{hc} - \epsilon_{cc}]\widehat{c}_t = \widehat{A}_t + \alpha(\widehat{k}_t - \widehat{h}_t) \tag{4.33}$$

$$E_t \widehat{A}_{t+1} = \rho \widehat{A}_t \tag{4.34}$$

$$\begin{aligned} \frac{1}{y}\widehat{d}_t = {} & \frac{1}{y}[\psi_1 d + 1 + r^*]\widehat{d}_{t-1} + s_c \widehat{c}_t \\ & + \frac{s_i}{\delta}[\widehat{k}_{t+1} - (1-\delta)\widehat{k}_t] \\ & - \widehat{A}_t - \alpha \widehat{k}_t - (1-\alpha)\widehat{h}_t \end{aligned} \tag{4.35}$$

$$\epsilon_{ch}\widehat{h}_t + \epsilon_{cc}\widehat{c}_t = \psi_1 \beta \widehat{d}_t + \epsilon_{ch} E_t \widehat{h}_{t+1} + \epsilon_{cc} E_t \widehat{c}_{t+1} \tag{4.36}$$

$$\begin{aligned} \epsilon_{cc}\widehat{c}_t + \epsilon_{ch}\widehat{h}_t + \Phi''(0)k(\widehat{k}_{t+1} - \widehat{k}_t) = {} & \epsilon_{cc} E_t \widehat{c}_{t+1} + \epsilon_{ch} E_t \widehat{h}_{t+1} \\ & + \frac{r^* + \delta}{1 + r^*}\left[E_t \widehat{A}_{t+1} + (\alpha - 1)(E_t \widehat{k}_{t+1} - E_t \widehat{h}_{t+1})\right] \\ & + \frac{\Phi''(0)k}{1 + r^*}[E_t \widehat{k}_{t+2} - E_t \widehat{k}_{t+1}] \end{aligned} \tag{4.37}$$

where $\epsilon_{hh} \equiv U_{hh}h/U_h$, $\epsilon_{ch} \equiv U_{ch}h/U_c$, $\epsilon_{hc} \equiv U_{hc}c/U_h$, $\epsilon_{cc} \equiv U_{cc}c/U_c$, $s_{tb} \equiv r^* d/F(k,h)$, $s_c \equiv c/F(k,h)$, $s_i \equiv \delta k/F(k,h)$, and $y \equiv F(k,h)$. The linearization uses the particular forms assumed for the production function and the country premium function. Of course, we could have linearized an expanded version of the equilibrium conditions, including equations defining additional macro indicators of interest. For instance, the system could have included equations (4.3), (4.4), (4.20), and (4.21), jointly defining y_t, i_t, tb_t, and ca_t.

We now express the set of equilibrium conditions and its linearized version using a more compact notation, which applies to a large class of dynamic stochastic general equilibrium models, not just the SOE-RBC model. Let y_t be a vector collecting the control variables of the model. Control variables in period t are endogenous variables that are determined in period t. In the SOE-RBC model, as defined by equations (4.11), (4.13), and (4.16)-(4.18) the vector y_t contains $\ln c_t$ and $\ln h_t$. Let x_t^1 denote the vector of endogenous state variables. Endogenous state variables in period t are endogenous variables determined before period t. In the SOE-RBC model, x_t^1 includes $\ln k_t$ and d_{t-1}. Let x_t^2 denote the vector of exogenous state variables. Exogenous state variables in period t are exogenous variables that are determined in period t or earlier. In the SOE-RBC model, x_t^2 includes a single varialbe, $\ln A_t$. Let $x_t \equiv [x_t^{1\prime} \quad x_t^{2\prime}]'$ denote the vector of state variables.

The equilibrium conditions of the model, given by equations (4.11), (4.13), and (4.16)-(4.18), can be written as

$$E_t f(y_{t+1}, y_t, x_{t+1}, x_t) = 0. \tag{4.38}$$

The law of motion of the exogenous state vector x_t^2 is given by

$$x_{t+1}^2 = \Lambda x_t^2 + \widetilde{\eta}\epsilon_{t+1}. \tag{4.39}$$

where, in general, ϵ_t is a vector of i.i.d. random variables with mean zero and unit variance, Λ is a square matrix with all eigenvalues inside the unit circle, and $\widetilde{\eta}$ is a matrix of parameters defining the variance covariance matrix of innovations to the exogenous state vector. In the SOE-RBC model ϵ_t. Λ, and $\widetilde{\eta}$ are all scalars (with $\Lambda = \rho$).

The deterministic steady state is a pair of constant vectors y and x that solves the system

$$f(y, y, x, x) = 0.$$

The steady-state vectors y and x are assumed to be known. In section 4.4, we derived the steady state of the SOE-RBC model analytically.

We restrict attention to equilibria in which at every date t the economy is expected to converge to the non-stochastic steady state, that is, we impose

$$\lim_{j\to\infty} \begin{bmatrix} E_t y_{t+j} \\ E_t x_{t+j} \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}, \tag{4.40}$$

This restriction implies that the transversality condition (4.19) is always satisfied.

As mentioned earlier, the representation of an equilibrium given by conditions (4.38)-(4.40) is quite general and applies to a large class of dynamic stochastic general equilibrium models. Thus, the solution technique discussed below is not restricted to the SOE-RBC model.

The first-oerder Taylor expansion of equation (4.38) is given by

$$f_{y'} E_t \widehat{y}_{t+1} + f_y \widehat{y}_t + f_{x'} E_t \widehat{x}_{t+1} + f_x \widehat{x}_t \tag{4.41}$$

where $\widehat{x}_t \equiv x_t - x$ and $\widehat{y}_t \equiv y_t - y$ denote, respectively, the deviations of x_t and y_t from their steady state values. The matrices $f_{y'}$, f_y, $f_{x'}$, and f_x denote, respectively, the partial derivatives of the function f with respect to y', y, x', and x evaluated at the nonstochastic steady state. These matrices are assumed to be known. Except for small models, like the SOE-RBC model studied here, these derivatives can be tedious to obtain by hand. The matlab scripts indicated at the end of this section perform and evaluate these derivatives automatically.

The solution of the linear system (4.41) with the associated exogenous law of motion (4.39), and the terminal condition (4.40) is given by

$$\widehat{x}_{t+1} = h_x \widehat{x}_t + \eta\, \epsilon_{t+1}$$

and

$$\widehat{y}_t = g_x \widehat{x}_t,$$

The matrix η is given by

$$\eta = \begin{bmatrix} \emptyset \\ \widetilde{\eta} \end{bmatrix}.$$

The Appendix shows how to obtain the matrices h_x and g_x, given the matrices $f_{y'}$, f_y, $f_{x'}$, and f_x. The Appendix also shows how to compute second moments and impulse response functions predicted by the model.

Matlab code for performing first-order accurate approximations to DSGE models and for computing second moments and impulse response functions is available at www.columbia.edu/~mu2166/1st_order.htm Matlab code to solve the specific SOE-RBC EDEIR model studied here is available online at www.columbia.edu/~mu2166/book/.

4.7 The Performance of the Model

Having calibrated the model and computed a first-order approximation to the equilibrium dynamics, we are ready to explore its quantitative predictions. As a point of reference, table 4.2 displays empirical second moments of interest from the Canadian economy. The first three columns display the empirical second moments reported by Mendoza (1991). The table shows standard deviations, serial correlations, and contemporaneous correlations of output with output, consumption, investment, hours, and the trade-balance-to-output ratio. The data is annual, quadratically detrended, and covers the period 1946-1985. Although outdated, we choose to use the empirical moments reported in Mendoza (1991) to preserve coherence with the calibration strategy of subsection 4.5. To gauge the stability of the empirical regularities and the out-of-sample performance of the SOE-RBC model, the middle three columns of table 4.2 display empirical second moments computed using data from 1960 to 2011. Overall, the stylized facts displayed in the table appear to be quite stable across time. In particular, in both samples the ranking of volatilities is $i > y > c > tb/y$. All aggregates are positively serially correlated. However, investment has become much more persistent over time. The trade-balance-to-GDP ratio is slightly countercyclical in the early sample but slightly procyclical in the recent one. The Canadian economy appears to have become more volatile, in particular, the volatilities of output and hours worked have increased by 30 and 80 percent, respectively.

Table 4.2 also displays second moments predicted by the the SOE-RBC EDEIR model. Comparing the early empirical second moments with their predicted counterparts, it should not come as a surprise that the model does very well at replicating the volatilities of output, hours, investment, and the trade-balance-to-output ratio, and the serial correlation of output. For we calibrated the parameters ω, ϕ, ψ_1, ρ, and $\widetilde{\eta}$ to match these five moments. But the model performs relatively well along other dimensions. For instance, it correctly implies that consumption is less volatile than output and investment and more volatile than hours and the trade-balance-to-output ratio. Also, the model correctly predicts that the trade balance-to-output ratio is countercyclical. This prediction is of interest because the parameters ϕ and ρ governing the degree of capital adjustment costs and the persistence of the productivity shock, which, as we established in the previous chapter, are key determinants of the cyclicality of the trade-balance-to-output ratio, were set independently of the observed cyclical properties of the trade balance. The model does not perform equally well at explaining the comovement of the trade balance with output over the more recent sample. Ex-

Table 4.2: Empirical and Theoretical Second Moments

Variable	Canadian Data						Model		
	1946 to 1985			1960 to 2011					
	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,GDP_t}	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,GDP_t}	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,GDP_t}
y	2.81	0.62	1	3.71	0.86	1	3.08	0.62	1
c	2.46	0.70	0.59	2.19	0.70	0.62	2.71	0.78	0.84
i	9.82	0.31	0.64	10.31	0.69	0.80	9.04	0.07	0.67
h	2.02	0.54	0.80	3.68	0.75	0.78	2.12	0.62	1
$\frac{tb}{y}$	1.87	0.66	-0.13	1.72	0.76	0.12	1.78	0.51	-0.04
$\frac{ca}{y}$							1.45	0.32	0.05

Note. Empirical moments for the peirod 1946 to 1985 are taken from Mendoza (1991) and for the period 1960 to 2011 are based on own calculations using data from WDI (GDP, consumption, investment, imports, and exports) and Statistics Canada (hours worked). All empirical second moments based on annual, per capita, and quadratically detrended data. Standard deviations are measured in percentage points. Theoretical moments are produced by running the Matlab code edeir_run.m.

ercise 4.8 asks you to use the empirical second moments associated with the 1960-2011 sample to recalibrate and evaluate the SOE-RBC EDEIR model.

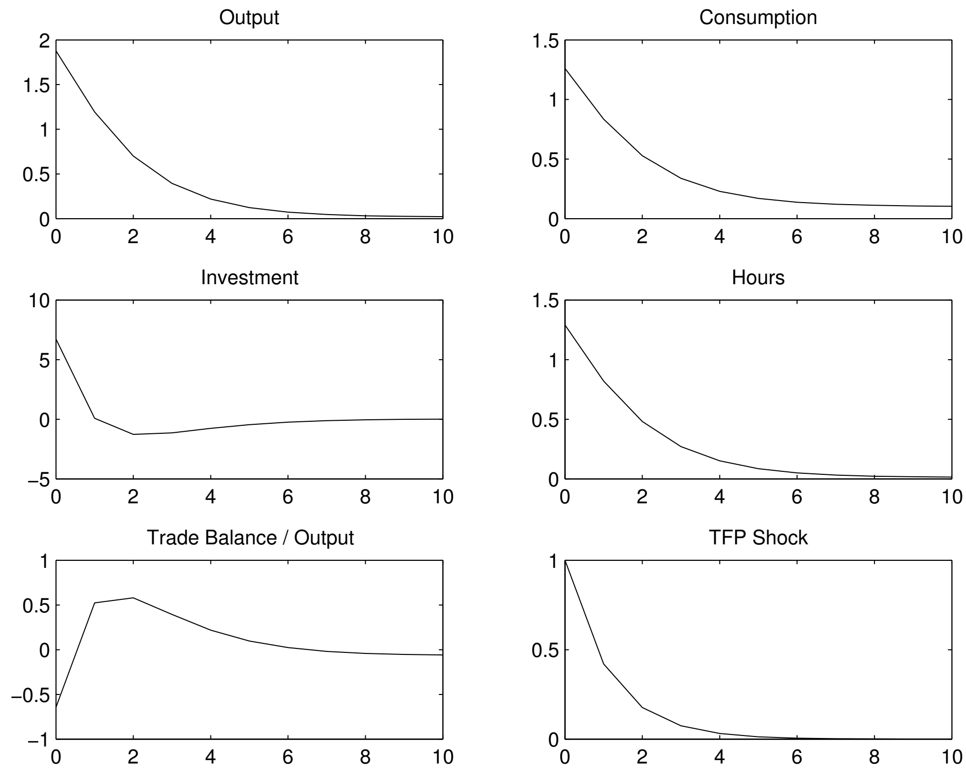
The model overpredicts the correlations of hours with output. In particular, the predicted correlation is exactly unity. This result is due to the assumed functional form for the period utility index. To see this, note that equilibrium condition (4.11), which equates the marginal product of labor to the marginal rate of substitution between consumption and leisure, can be written as $h_t^\omega = (1-\alpha)y_t$. The log-linearized version of this condition is $\omega\widehat{h}_t = \widehat{y}_t$, which implies that $\widehat{h}_t$ and $\widehat{y}_t$ are perfectly correlated.

Figure 4.1. displays the impulse response functions of a number of variables of interest to a technology shock of size 1 percent in period 0. In response to this innovation, the model predicts an expansion in output, consumption, investment, and hours and a deterioration in the trade-balance-to-output ratio. The level of the trade balance, not shown, also falls on impact. This means that the initial increase in domestic absorption (i.e., the increase in $c_0 + i_0$) is larger than the increase in output. Further, the initial response of consumption is proportionally smaller than that of output, whereas the initial response of investment is about eight times as large as that of output. It follows that in the context of the present SOE-RBC model investment plays a key role in generating a countercyclical initial response of the trade balance.

4.8 The Role of Persistence and Capital Adjustment Costs

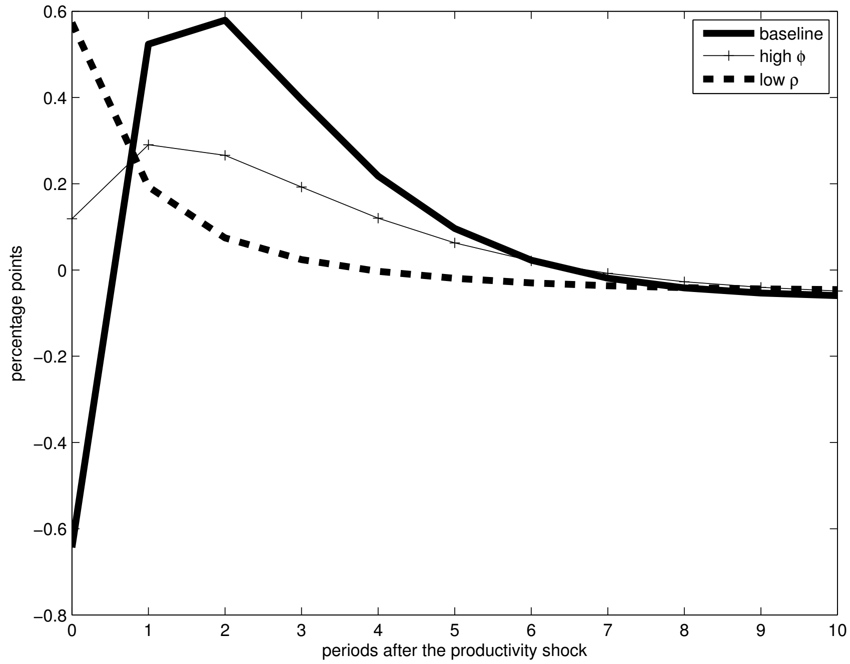
In the previous chapter, we deduced that the negative response of the trade balance to a positive technology shock was not a general implication of the neoclassical model. In particular, Principles I and II of the previous chapter state that two conditions must be met for the model to generate a deterioration in the external accounts in response to a mean-reverting improvement in total factor productivity. First, capital adjustment costs must not be too stringent. Second, the productivity

Figure 4.1: Responses to a One-Percent Productivity Shock



Note. To produce this figure, run the Matlab code edeir_run.m.

Figure 4.2: Response of the Trade-Balance-To-Output Ratio to a Positive Technology Shock



shock must be sufficiently persistent. To illustrate this conclusion, figure 4.2 displays the impulse response function of the trade balance-to-GDP ratio to a technology shock of unit size in period 0 under three alternative parameter specifications. The solid line reproduces the benchmark case from figure **??**. The broken line depicts an economy where the persistence of the productivity shock is half as large as in the benchmark economy ($\rho = 0.21$). In this case, because the productivity shock is expected to die out quickly, the response of investment is relatively weak. In addition, the temporariness of the shock induces households to save most of the increase in income to smooth consumption over time. As a result, the expansion in aggregate domestic absorption is modest. At the same time, because the size of the productivity shock is the same as in the benchmark economy, the initial responses of output and hours are identical in both economies (recall that, by equation (4.32), h_t depends only on k_t and A_t, and that k_t is predetermined in period t). The combination of a weak response in domestic absorption and an initial response in output that is independent of the value of ρ, results in an improvement in the trade balance when productivity shocks are not too persistent.

The crossed line depicts the case of high capital adjustment costs. Here the parameter ϕ equals 0.084, a value three times as large as in the benchmark case. In this environment, high adjustment costs discourage firms from increasing investment spending by as much as in the benchmark economy. As a result, the response of aggregate domestic demand is weaker, leading to an improvement in the trade balance-to-output ratio.

4.9 The SOE-RBC Model With Complete Asset Markets (CAM)

The SOE-RBC model economy considered thus far features incomplete asset markets. In that model, agents have access to a single financial asset that pays a non-state-contingent rate of return. In the model studied in this section, by contrast, agents are assumed to have access to a complete array of state-contingent claims. As we will see, the introduction of complete asset markets per se induces stationarity in the equilibrium dynamics, so there will be no need to introduce any ad-hoc stationarity inducing feature.

Preferences and technologies are as in the EDEIR model. The period-by-period budget constraint of the household is given by

$$E_t r_{t,t+1} b_{t+1} = b_t + A_t F(k_t, h_t) - c_t - [k_{t+1} - (1-\delta)k_t] - \Phi(k_{t+1} - k_t), \tag{4.42}$$

where $r_{t,t+1}$ is a pricing kernel such that the period-t price of a random payment b_{t+1} in period $t+1$ is given by $E_t r_{t,t+1} b_{t+1}$. To clarify the nature of the pricing kernel $r_{t,t+1}$, define the current state of nature as S^t. Let $p(S^{t+1}|S^t)$ denote the price of a contingent claim that pays one unit of consumption in a particular state S^{t+1} following the current state S^t. Then the current price of a portfolio composed of $b(S^{t+1}|S^t)$ units of contingent claims paying in states S^{t+1} following S^t is given by $\sum_{S^{t+1}|S^t} p(S^{t+1}|S^t)b(S^{t+1}|S^t)$. Now let $\pi(S^{t+1}|S^t)$ denote the probability of occurrence of state S^{t+1}, given information available at the current state S^t. Multiplying and dividing the expression inside the summation sign by $\pi(S^{t+1}|S^t)$ we can write the price of the portfolio as $\sum_{S^{t+1}|S^t} \pi(S^{t+1}|S^t)\frac{p(S^{t+1}|S^t)}{\pi(S^{t+1}|S^t)}b(S^{t+1}|S^t)$. Now let $r_{t,t+1} \equiv p(S^{t+1}|S^t)/\pi(S^{t+1}|S^t)$ be the price of a contingent claim that pays in state $S^{t+1}|S^t$ scaled by the inverse of the probability of occurrence of the state in which the claim pays. Also, let $b_{t+1} \equiv b(S^{t+1}|S^t)$. Then, we can write the price of the portfolio as $\sum_{S^{t+1}|S^t} \pi(S^{t+1}|S^t)r_{t,t+1}b_{t+1}$. But this expression is simply the conditional expectation $E_t r_{t,t+1} b_{t+1}$.

Note that $E_t r_{t,t+1}$ is the price in period t of an asset that pays 1 unit of consumption goods in every state of period $t+1$. It follows that

$$1 + r_t \equiv \frac{1}{E_t r_{t,t+1}}$$

represents the risk-free real interest rate in period t.

Households are assumed to be subject to a no-Ponzi-game constraint of the form

$$\lim_{j \to \infty} E_t r_{t,t+j} b_{t+j} \geq 0, \tag{4.43}$$

at all dates and under all contingencies. The variable

$$r_{t,t+j} \equiv r_{t,t+1} r_{t+1,t+2} \cdots r_{t+j-1,t+j}$$

represents the pricing kernel such that $E_t r_{t,t+j} b_{t+j}$ is the period-t price of a stochastic payment

b_{t+j} in period $t+j$. Clearly, $r_{t,t}=1$.

To characterize the household's optimal plan, it is convenient to derive an intertemporal budget constraint. Begin by multiplying both sides of the sequential budget constraint (4.42) by $r_{0,t}$. Then apply the conditional expectations operator E_0 to obtain

$$E_0 r_{0,t} E_t r_{t,t+1} b_{t+1} = E_0 r_{0,t} \left[b_t + A_t F(k_t, h_t) - c_t - k_{t+1} + (1-\delta) k_t - \Phi(k_{t+1} - k_t) \right].$$

By the definition of the pricing kernel and the law of iterated expectations, we have that $E_0 r_{0,t} E_t r_{t,t+1} b_{t+1} = E_0 r_{0,t+1} b_{t+1}$. So we can write the above expression as

$$E_0 r_{0,t+1} b_{t+1} = E_0 r_{0,t} \left[b_t + A_t F(k_t, h_t) - c_t - k_{t+1} + (1-\delta) k_t - \Phi(k_{t+1} - k_t) \right].$$

Now sum this expression for $t=0$ to $t=T>0$. This yields

$$E_0 r_{0,T+1} b_{T+1} = b_0 + E_0 \sum_{t=0}^{T} r_{0,t} \left[A_t F(k_t, h_t) - c_t - k_{t+1} + (1-\delta) k_t - \Phi(k_{t+1} - k_t) \right].$$

Take limit for $T \to \infty$ and use the no-Ponzi-game constraint (4.43) to obtain

$$b_0 \geq E_0 \sum_{t=0}^{\infty} r_{0,t} \left[c_t + k_{t+1} - (1-\delta) k_t + \Phi(k_{t+1} - k_t) - A_t F(k_t, h_t) \right]. \tag{4.44}$$

This expression states that the period-0 value of the stream of current and future trade deficits cannot exceed the value of the initial asset position b_0.

The household's problem consists in choosing contingent plans $\{c_t, h_t, k_{t+1}\}$ to maximize the lifetime utility function (4.1) subject to (4.44), given k_0, b_0, and exogenous processes $\{A_t, r_{0,t}\}$.

The Lagrangian associated with this problem is

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \left\{ \beta^t U(c_t, h_t) + \xi_0 r_{0,t} \left[A_t F(k_t, h_t) - c_t - k_{t+1} + (1-\delta)k_t - \Phi(k_{t+1} - k_t) \right] \right\} + \xi_0 b_0,$$

where $\xi_0 > 0$ denotes the Lagrange multiplier on the time-0 present-value budget constraint (4.44). The first-order conditions associated with the household's maximization problem are (4.11), (4.18), (4.44) holding with equality, and

$$\beta^t U_c(c_t, h_t) = \xi_0 r_{0,t}. \tag{4.45}$$

Taking the ratio of this expression to itself evaluated in period $t+1$ yields

$$\frac{\beta U_c(c_{t+1}, h_{t+1})}{U_c(c_t, h_t)} = r_{t,t+1},$$

which says that consumers equate their intertemporal marginal rate of substitution of current consumption for consumption in a particular state next period to the price of the corresponding state contingent claim scaled by the probability of occurrence of that state. Rearranging the above expression, taking expectations conditional on information available in period t, and recalling the definition of the risk-free interest rate given above yields

$$U_c(c_t, h_t) = \beta(1 + r_t) E_t U_c(c_{t+1}, h_{t+1}),$$

which is identical to equation (4.7) in the incomplete asset market version of the model. In other words, the complete asset market model generates a state-by-state version of the Euler equation implied by the incomplete-asset-market model, reflecting the fact that in the present environment consumers have more financial instruments available to diversify risk.

We assume that the economy is small and fully integrated to the international financial market.

Let $r^*_{0,t}$ denote the pricing kernel prevailing in international financial markets. By the assumption of free capital mobility, we have that domestic asset prices must be equal to foreign asset prices, that is,

$$r_{0,t} = r^*_{0,t} \tag{4.46}$$

for all dates and states. Foreign households are also assumed to have unrestricted access to international financial markets. Therefore, a condition like (4.45) must also hold abroad. Formally,

$$\beta^t U^*_{c^*}(c^*_t, h^*_t) = \xi^*_0 r^*_{0,t}. \tag{4.47}$$

Note that we are assuming that domestic and foreign households share the same subjective discount factor, β. Combining (4.45)-(4.47) yields

$$U_c(c_t, h_t) = \frac{\xi_0}{\xi^*_0} U^*_{c^*}(c^*_t, h^*_t)$$

for all dates and states. This expression says that under complete asset markets, the marginal utility of consumption is perfectly correlated across countries. The ratio $\frac{\xi_0}{\xi^*_0}$ reflects differences in per capita wealth between the domestic economy and the rest of the world. Because the present model is one of a small open economy, c^*_t and h^*_t are taken as exogenously given. We endogenize the determination of c^*_t and h^*_t in exercise 4.9 at the end of this chapter. This exercise analyzes a two-country model with complete asset markets in which one country is large and the other is small.

Because the domestic economy is small, the domestic productivity shock A_t does not affect the foreign variables, which respond only to foreign shocks. The domestic economy, however, can be affected by foreign shocks via c^*_t and h^*_t. To be in line with the stochastic structure of the EDEIR model, we shut down all foreign shocks and focus attention only on the effects of innovations in domestic productivity. Therefore, we assume that the foreign marginal utility of consumption is time invariant and given by $U_c^*(c^*, h^*)$, where c^* and h^* are constants. Let $\psi_{cam} \equiv \frac{\xi_0}{\xi_0^*} U_{c^*}^*(c^*, h^*)$. Then, we can rewrite the above expression as

$$U_c(c_t, h_t) = \psi_{cam}. \tag{4.48}$$

This expression reflects the fact that, because domestic consmers have access to a complete set of Arrow-Debreu contingent assets, they can fully diversify domestic risk. Thus, domestic consumers are exposed only to aggregate external risk. We are assuming that aggregate external risk is nil. As a result, by appropriately choosing their asset portfolios, domestic consumers can attain a constant marginal utility of consumption at all times and under all contingencies. Exercise 4.3 at the end of this chapter studies a version of the present model in which ψ_{cam} is stochastic, reflecting the presence of external shocks.

The competitive equilibrium of the CAM economy is a set of processes $\{c_t, h_t, k_{t+1}, A_t\}$ satisfying (4.11), (4.12), (4.18), and (4.48), given A_0, k_0, and the exogenous process $\{\epsilon_t\}$.

The CAM model delivers starionary processes for all variables of interest. This means that replacing the assumption of incomplete asset markets for the assumption of complete asset markets eliminates the endogenous random walk problem that plagues the dynamics of the one-bond economy. The key feature of the complete asset market responsible for its stationarity property is equation (4.48), which states that with complete asset markets the marginal utility of consumption is constant. By contrast, in the one-bond model, in the absence of any ad-hoc stationarity inducing feature, the marginal utility of consumption follows a random walk. To see this, set $\beta(1 + r_t) = 1$ for all t in equation (4.7).

We now wish to shed light on a question that arises often in models with complete asset markets, namely, what is the current account when financial markets are complete? In the one-bond economy the answer is simple: the current account can be measured either by changes in net holdings of the internationally traded bond or by the sum of the trade balance and net interest income paid by the single bond. Under complete asset markets, there is a large (possibly infinite) number of state-contingent financial assets, each with different returns. As a result, it is less clear how to keep track of the country's net foeign asset position or of its net investment income. It turns out that there is a simple way of characterizing and computing the equilibrium level of the current account. Let us begin by addressing the simpler question of defining the trade balance. As in the one-bond model, the trade balance in the CAM model is simply given by equation (4.20). The current account can be defined as the change in the country's net foreign asset position. Let

$$s_t \equiv E_t r_{t,t+1} b_{t+1}$$

denote the net foreign asset position at the end of period t. Then, the current account is given by

$$ca_t = s_t - s_{t-1}.$$

Alternatively, the current account can be expressed as the sum of the trade balance and net investment income. In turn, net investment income is given by the difference between the payoff in period t of assets acquired in $t-1$, given by b_t, and the resources spent in $t-1$ on purchases of contingent claims, given by $E_{t-1} r_{t-1,t} b_t$. Thus, the current account is given by

$$ca_t = tb_t + b_t - E_{t-1} r_{t-1,t} b_t.$$

To see that the above two definitions of the current account are identical, use the definition of the trade balance, equation (4.20), and the definition of the net foreign asset position s_t to write the

Table 4.3: The SOE-RBC Model With Complete Asset Markets: Predicted Second Moments

Variable	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,GDP_t}
y	3.1	0.61	1.00
c	1.9	0.61	1.00
i	9.1	0.07	0.66
h	2.1	0.61	1.00
$\frac{tb}{y}$	1.6	0.39	0.13
$\frac{ca}{y}$	3.1	-0.07	-0.49

Note. Standard deviations are measured in percentage points. Matlab code to produce this table is available at `http://www.columbia.edu/~mu2166/closing.htm`.

sequential resource constraint (4.42) as

$$s_t = tb_t + b_t$$

Subtracting s_{t-1} from both sides of this expression, we have

$$s_t - s_{t-1} = tb_t + b_t - E_{t-1} r_{t-1,t} b_t.$$

The left-hand side of this expression is our first definition of the current account, and the right-hand side our second definition.

The functions U, F, and Φ are parameterized as in the EDEIR model. The parameters σ, β, ω, α, ϕ, δ, ρ, and $\widetilde{\eta}$ take the values displayed in table 4.1. The parameter ψ_{cam} is set so as to ensure that the steady-state levels of consumption in the CAM and EDEIR models are the same.

Table 4.3 displays unconditional second moments predicted by the SOE-RBC model with complete asset markets. The predictions of the model regarding output, consumption, investment, and the trade balance are qualitatively similar to those of the (EDEIR) incomplete-asset-market model. In particular, the model preserves the volatility ranking of output, consumption, investment, and the trade balance. Also, the domestic components of aggregate demand are all positively serially correlated and procyclical. Note that the correlation of consumption with output is now unity. This prediction of the CAM model is a consequence of assuming complete markets and GHH preferences. Under complete asset markets the marginal utility of consumption is constant over time, so that up to first order consumption is linear in hours. In turn, with GHH preferences, as we deduced earlier in this chapter, hours are linearly related to output up to first order. A significant difference between the predictions of the complete- and incomplete-asset-market models is that the former implies a highly countercyclical current account, whereas the latter implies an acyclical current account.

4.10 Alternative Ways to Induce Stationarity

The small open economy RBC model analyzed thus far features a debt-elastic country interest-rate premium. As mentioned earlier in this chapter, the inclusion of a debt-elastic premium responds to the need to obtain stationary dynamics up to first order. Had we assumed a constant interest rate, the linearized equilibrium dynamics would have contained an endogenous random walk component and the steady state would have depended on initial conditions. Two problems emerge when the linear approximation possesses a unit root. First, one can no longer claim that when the support of the underlying shocks is sufficiently small the linear system behaves like the original nonlinear system, which is ultimately the focus of interest. Second, when the variables of interest contain random walk elements, it is impossible to compute unconditional first and second moments, such as standard deviations, serial correlations, correlations with output, etc., which are the most common descriptive statistics of the business cycle.

Nonstationarity arises in the small open economy model from three features: an exogenous cost of borrowing in international financial markets, an exogenous subjective discount factor, and incomplete asset markets. Accordingly, in this section we study stationarity inducing devises that consist in altering one of these three features. Our analysis follows closely Schmitt-Grohé and Uribe (2003), but expands their analysis by including two additional approaches to inducing stationarity, a model with an internal interest-rate premium, a model with perpetually-young consumers, and a model in which stationarity is induced by approximating the equilibrium using global methods making agents slightly impatient by making the discount facor β smaller than the pecuniary discount factor $1/(1 + r^*)$.

One important question is whether the different stationarity inducing devises affect the predicted business cycle of the small open economy. A result of this section is that, given a commong calibration, all models considered deliver similar business cycles.

Before plunging into details, it is important to note that the nature of the non-stationarity that is present in the small open economy model is different from the one that emerges from the introduction of non-stationary exogenous shocks. In the latter case, it is typically possible to find a transformation of variables that renders the model economy stationary in terms of the transformed variables. We will study an economy with non-stationary shocks and provide an example of a stationarity inducing transformation in section 5.2 of chapter 5. By contrast, the nonstationarity that arises in the small open economy model with an exogenous cost of borrowing, an exogenous rate of time preference, and incomplete markets cannot be eliminated by any variable transformations.

The section proceeds by first presenting and calibrating the different stationarity inducing theoretical devises and then comparing the quantitative predictions of the various models.

4.10.1 Internal Debt-Elastic Interest Rate (IDEIR)

The EDEIR model studied thus far assumes that the country interest-rate premium depends upon the cross-sectional average of external debt. As a result, households take the country premium as exogenously given. The model with an internal debt-elastic interest rate assumes instead that the interest rate faced by domestic agents is increasing in the individual debt position, d_t. Consequently, households internalize the effect that their borrowing choices have on the interest rate they face. In all other aspects, the IDEIR and EDEIR models are identical.

Formally, in the IDEIR model the interest rate is given by

$$r_t = r^* + p(d_t), \tag{4.49}$$

where r^*, as before, denotes the world interest rate, but now $p(\cdot)$ is a household-specific interest-rate premium. Note that the argument of the interest-rate premium function is the household's own net debt position. This means that in deciding its optimal expenditure and savings plan, the household will take into account the fact that a change in its debt position alters the marginal cost of funds. The only optimality condition that changes relative to the EDEIR model is the Euler equation for debt accumulation, which now takes the form

$$U_c(c_t, h_t) = \beta[1 + r^* + p(d_t) + p'(d_t)d_t]E_t U_c(c_{t+1}, h_{t+1}). \tag{4.50}$$

This expression features the derivative of the premium with respect to debt because households internalize the fact that as their net debt increases, so does the interest rate they face in financial markets. As a result, in the margin, the householdl cares about the marginal cost of borrowing $1 + r^* + p(d_t) + p'(d_t)d_t$ and not about the average cost of borrowing, $1 + r^* + p(d_t)$.

The competitive equilibrium of the IDEIR economy is a set of processes $\{d_t, c_t, h_t, k_{t+1}, A_t\}$

satisfying (4.11), (4.12), (4.16), (4.18), (4.19), and (4.50), given A_0, d_{-1}, and k_0, and the process $\{\epsilon_t\}$.

We assume the same functional forms and parameter values as in the EDEIR model (see section 4.3). We note that in the model analyzed here the steady-state level of debt is no longer equal to $\overline{d}$. To see this, recall that $\beta(1+r^*) = 1$ and note that the steady-state version of equation (4.50) imposes the following restriction on d,

$$(1+d)e^{d-\overline{d}} = 1,$$

which does not admit the solution $d = \overline{d}$, except in the special case in which $\overline{d} = 0$. We set $\overline{d} = 0.7442$, which is the value imposed in the EDEIR model. The implied steady-state level of debt is then given by $d = 0.4045212$. Intuitively, households internalize that their own debt position drives up the interest rate, hence they choose to borrow less than households in the EDEIR economy, who fail to internalize the dependence of the interest rate on the stock of debt. In this sense, one can say that households in the EDEIR economy overborrow. The fact that the steady-state debt is lower than $\overline{d}$ implies that the country premium is negative in the steady state. However, the marginal country premium, given by $p(d_t) + p'(d_t)d_t$, is nil in the steady state, as it is in the EDEIR economy. Recall that in the EDEIR economy, the marginal and average premia perceived by households are equal to each other and given by $p(\widetilde{d}_t)$. An alternative calibration strategy is to impose $d = \overline{d}$, and to adjust β to ensure that equation (4.50) holds in the deterministic steady state. In this case, the country premium vanishes in the steady state, but the marginal premium is positive and equal to $\psi_1\overline{d}$.

4.10.2 Portfolio Adjustment Costs (PAC)

In the portfolio adjustment cost (PAC) model, stationarity is induced by assuming that agents face convex costs of holding assets in quantities different from some long-run level. Preferences and technology are as in the EDEIR model. However, in contrast to what is assumed in that model, in the PAC model the interest rate at which domestic households can borrow from the rest of the world is assumed to be constant and equal to the world interest rate, r^*, that is, the country premium is nil at all times. The sequential budget constraint of the household is given by

$$d_t = (1+r^*)d_{t-1} - A_t F(k_t, h_t) + c_t + k_{t+1} - (1-\delta)k_t + \Phi(k_{t+1} - k_t) + \Psi(d_t), \tag{4.51}$$

where $\Psi(\cdot)$ is a convex portfolio adjustment cost function satisfying $\Psi(\overline{d}) = \Psi'(\overline{d}) = 0$, for some $\overline{d}$. The first-order conditions associated with the household's maximization problem are identical to those associated with the EDEIR model, except that the Euler condition for debt, equation (4.17), now becomes

$$U_c(c_t, h_t) = \beta \frac{1+r^*}{1-\Psi'(d_t)} E_t U_c(c_{t+1}, h_{t+1}). \tag{4.52}$$

This optimality condition implies that the effective interest rate faced by the household, which we denote r_t, is debt elastic and given by

$$1 + r_t = \frac{1+r^*}{1-\Psi'(d_t)}. \tag{4.53}$$

Because the portfolio adjustment cost function is convex, the effective interest rate is increasing in the stock of debt. In this regard, the PAC model is a close relative of the EDEIR model, as can be seen by comparing the above Euler equation with its counterpart in the EDEIR model, given by equation (4.17).

The specification adopted here assumes that households directly borrow from abroad. This setup can be decentralized as follows. Suppose that households face no portfolio adjustment costs and can borrow and lend at the interest rate r_t, which they take as exogenously given and, in particular, as independent of their own debt positions. Their sequential budget constraint is then given by $\tilde{d}_t = (1 + r_{t-1})\tilde{d}_{t-1} - A_t F(k_t, h_t) + c_t + k_{t+1} - (1 - \delta)k_t + \Phi(k_{t+1} - k_t) - \Pi_t$, where $\tilde{d}_t$ denotes household debt in period t and Π_t denotes profit income in period t, which the household takes as exogenously given. The optimality conditions associated with the household problem are identical to those in centralized version of the model, except that the Euler equation now becomes $U_c(c_t, h_t) = \beta(1 + r_t)E_t U_c(c_{t+1}, h_{t+1})$.

Assume that financial transactions between domestic and foreign residents are intermediated by domestic financial institutions, or banks. Suppose that there is a continuum of banks of measure one that behave competitively. They capture funds, d_t, from foreign investors at the world interest rate r^* and lend $\tilde{d}_t$ to domestic agents at the interest rate r_t. Banks face operational costs, $\Psi(d_t)$, that are increasing and convex in the volume of intermediation, d_t. Bank profits in period $t+1$ are given by $\Pi_{t+1} \equiv (1 + r_t)\tilde{d}_t - (1 + r^*)d_t$. Banks are subject to the resource constraint $\tilde{d}_t = d_t - \Psi(d_t)$. The problem of domestic banks is then to choose $\tilde{d}_t$ and d_t to maximize profits subject to the resource constraint, taking r_t as given. The first-order condition associated with the bank's profit maximization problem is $1 + r_t = \frac{1+r^*}{1-\Psi'(d_t)}$, which is identical to equation (4.53). Each period bank profits are distributed to domestic households in a lump-sum fashion. Replacing the expression for bank profits in the household's budget constraint and using the bank's resource constraint yields the budget constraint of the centralized economy, equation (4.51). It follows that the equilibrium allocations of the centralized and the decentralized economies are the same.

The competitive equilibrium of the PAC economy is a set of processes $\{d_t, c_t, h_t, k_{t+1}, A_t\}$ satisfying (4.11), (4.12), (4.18), (4.19), (4.51), and (4.52), given A_0, d_{-1}, and k_0, and the process $\{\epsilon_t\}$.

The world interest rate is assumed to satisfy

$$\beta(1+r^*) = 1.$$

This assumption implies that in the steady state, the Euler equation (4.52) becomes

$$\Psi'(d) = 0,$$

where d denotes the steady-state value of debt. The assumptions imposed on the portfolio adjustment cost $\Psi(\cdot)$ imply that the unique solution to the above expression is $d = \overline{d}$. It follows that the steady-state level of debt is independnet of initial conditions.

We assume a quadratic form for $\Psi(\cdot)$,

$$\Psi(d_t) = \frac{\psi_2}{2}(d_t - \overline{d})^2,$$

where ψ_2 and $\overline{d}$ are constant parameters defining the portfolio adjustment cost function. The remaining functional forms and the calibration of common parameters are as in the EDEIR model. We calibrate $\overline{d}$ to 0.7442, which is the same value as in the EDEIR model. This means that the steady-state values of all endogenous variables are the same in the PAC and EDEIR models. We set ψ_2 at 0.00074, which ensures that the volatility of the current-account-to-output ratio is the same as in the EDEIR model.

At this point, it might be natural to expect the analysis of an external version of the PAC model in which the portfolio adjustment cost depends on the aggregate level of debt, $\widetilde{d}_t$, as opposed to the individual debt position d_t. However, this modification would fail to render the small open economy model stationary. The reason is that in this case, the optimality condition with respect to debt, given by equation (4.52) in the PAC model, would become $U_c(c_t, h_t) = \beta(1 + r^*)E_t U_c(c_{t+1}, h_{t+1})$, which, because $\beta(1+r^*)$ equals one, implies that the marginal utility of consumption follows a random walk, and is therefore nonstationary.

4.10.3 External Discount Factor (EDF)

We next study an SOE RBC model in which stationarity is induced by assuming that the subjective discount factor depends upon endogenous variables. Specifically, we consider a preference specification in which the discount factor depends on endogenous variables that are taken as exogenous by individual households. We refer to this environment as the external discount factor (EDF) model.

Suppose that the discount factor depends on the average per capita levels of consumption and hours worked. Formally, preferences are described by

$$E_0 \sum_{t=0}^{\infty} \theta_t U(c_t, h_t) \tag{4.54}$$

$$\theta_{t+1} = \beta(\widetilde{c}_t, \widetilde{h}_t)\theta_t \quad t \geq 0, \quad \theta_0 = 1; \tag{4.55}$$

where $\widetilde{c}_t$ and $\widetilde{h}_t$ denote the cross-sectional averages of per capita consumption and hours, respectively, which the individual household takes as exogenously given.

In the EDF model, the interest rate is assumed to be constant and equal to r^*. The sequential budget constraint of the household therefore takes the form

$$d_t = (1+r^*)d_{t-1} - A_t F(k_t, h_t) + c_t + k_{t+1} - (1-\delta)k_t + \Phi(k_{t+1} - k_t), \tag{4.56}$$

and the no-Ponzi-game constraint simplifies to $\lim_{j\to\infty}(1+r^*)^{-j}E_t d_{t+j} \leq 0$.

The first-order conditions associated with the household's maximization problem are (4.11), (4.56), and

$$U_c(c_t, h_t) = \beta(\widetilde{c}_t, \widetilde{h}_t)(1+r^*)E_t U_c(c_{t+1}, h_{t+1}) \tag{4.57}$$

$$U_c(c_t,h_t)[1+\Phi'(k_{t+1}-k_t)] = \beta(\widetilde{c}_t,\widetilde{h}_t)E_tU_c(c_{t+1},h_{t+1})\left[A_{t+1}F_k(k_{t+1},h_{t+1})+1-\delta+\Phi'(k_{t+2}-k_{t+1})\right] \tag{4.58}$$

$$\lim_{j\to\infty} E_t\frac{d_{t+j}}{(1+r^*)^j} = 0. \tag{4.59}$$

In equilibrium, individual and average per capita levels of consumption and effort are identical. That is,

$$c_t = \widetilde{c}_t \tag{4.60}$$

and

$$h_t = \widetilde{h}_t. \tag{4.61}$$

A competitive equilibrium is a set of processes $\{d_t,\, c_t,\, h_t,\, \widetilde{c}_t,\, \widetilde{h}_t,\, k_{t+1},\, A_t\}$ satisfying (4.11), (4.12), and (4.56)-(4.61), given A_0, d_{-1}, and k_0 and the stochastic process $\{\epsilon_t\}$.

We evaluate the model using the same functional forms for the period utility function, the production function, and the capital adjustment cost function as in the EDEIR model. We assume that the subjective discount factor is of the form

$$\beta(c,h) = \left(1 + c - \frac{h^\omega}{\omega}\right)^{-\psi_3},$$

with $\psi_3 > 0$, so that increases in consumption or leisure make households more impatient.

To see that in the EDF model the steady-state level of debt is determined independently of initial conditions, start by noticing that in the steady state, equation (4.57) implies that

$$\beta(c,h)(1+r^*) = 1,$$

where c and h denote the steady-state values of consumption and hours. Next, notice that, given this result, the steady-state values of hours, capital (k), and output ($k^\alpha h^{1-\alpha}$) can be found in exactly the same way as in the EDEIR model, with β replaced by $(1+r^*)^{-1}$. Notice that k and h depend only on the deep structural parameters r^*, α, ω, and δ. With h in hand, the above expression delivers c, which depends only on the deep structural parameters defining h and on ψ_3. Finally, in the steady state, the resource constraint (4.56) implies that the steady state level of debt, d, is given by $d = (c + \delta k - k^\alpha h^{1-\alpha})/r^*$, which depends only on structural parameters.

The EDF model features one new parameter relative to the EDEIR model, namely the elasticity of the discount factor relative to the composite $1 + c_t - h_t^\omega/\omega$. We set ψ_3 to ensure that the steady-state trade-balance-to-output ratio equals 2 percent, in line with the calibration of the EDEIR model. The implied value of ψ_3 is 0.11.

Note that in our assumed specification of the endogenous discount factor, the parameter ψ_3 governs both the steady-state trade-balance-to-output ratio and the stationarity of the equilibrium dynamics. This dual role may create a conflict. On the one hand, one may want to set ψ_3 at a small value so as to ensure stationarity without affecting the predictions of the model at business-cycle frequency. On the other hand, matching the observed average trade-balance-to-output ratio might require a value of ψ_3 that does affect the behavior of the model at business-cycle frequency. For this reason, it might be useful to consider a two-parameter specification of the discount factor, such as $\beta(c_t, h_t) = (\widetilde{\psi}_3 + c_t - \omega^{-1} h_t^\omega)^{-\psi_3}$, where $\widetilde{\psi}_3 > 0$ is a parameter. With this specification, one can fix the parameter ψ_3 at a small value, just to ensure stationarity, and set the parameter $\widetilde{\psi}_3$ to match the observed trade-balance-to-output ratio.

4.10.4 Internal Discount Factor (IDF)

Consider now a variation of the EDF model in which the subjective discount factor depends on the individual levels of consumption and hours worked rather than on the aggregate levels. Specifically, suppose that preferences are given by equation (4.54), with the following law of motion for θ_t:

$$\theta_{t+1} = \beta(c_t, h_t)\theta_t \quad t \geq 0, \quad \theta_0 = 1. \tag{4.62}$$

This preference specification was conceived by Uzawa (1968) and introduced in the small-open-economy literature by Mendoza (1991). Under these preferences, households internalize that their choices of consumption and leisure affect their valuations of future period utilities.

Households choose processes $\{c_t, h_t, k_{t+1}, d_t, \theta_{t+1}\}_{t=0}^{\infty}$ so as to maximize the utility function (4.54) subject to the sequential budget constraint (4.56), the law of motion of the discount factor (4.62), and the same no-Ponzi constraint as in the EDF economy. Let $\theta_t\lambda_t$ denote the Lagrange multiplier associated with (4.56) and $\theta_t\eta_t$ the Lagrange multiplier associated with (4.62). The first-order conditions associated with the household's maximization problem are (4.56), (4.59), and

$$\lambda_t = \beta(c_t, h_t)(1 + r_t)E_t\lambda_{t+1} \tag{4.63}$$

$$U_c(c_t, h_t) - \eta_t\beta_c(c_t, h_t) = \lambda_t \tag{4.64}$$

$$-U_h(c_t, h_t) + \eta_t\beta_h(c_t, h_t) = \lambda_t A_t F_h(k_t, h_t) \tag{4.65}$$

$$\eta_t = -E_t U(c_{t+1}, h_{t+1}) + E_t\eta_{t+1}\beta(c_{t+1}, h_{t+1}) \tag{4.66}$$

$$\lambda_t[1 + \Phi'(k_{t+1} - k_t)] = \beta(c_t, h_t)E_t\lambda_{t+1}\left[A_{t+1}F_k(k_{t+1}, h_{t+1}) + 1 - \delta + \Phi'(k_{t+2} - k_{t+1})\right] \tag{4.67}$$

These first-order conditions are fairly standard, except for the fact that the marginal utility of consumption is not given simply by $U_c(c_t, h_t)$ but rather by $U_c(c_t, h_t) - \beta_c(c_t, h_t)\eta_t$. The second term in this expression reflects the fact that an increase in current consumption lowers the discount factor ($\beta_c < 0$). In turn, a unit decline in the discount factor reduces utility in period t by η_t. Intuitively,

$-\eta_t$ equals the expected present discounted value of utility from period $t+1$ onward. To see this, iterate the first-order condition (4.66) forward to obtain $\eta_t = -E_t \sum_{j=1}^{\infty} \left(\frac{\theta_{t+j}}{\theta_{t+1}}\right) U(c_{t+j}, h_{t+j})$. Similarly, the marginal disutility of labor is not simply $U_h(c_t, h_t)$ but instead $U_h(c_t, h_t) - \beta_h(c_t, h_t)\eta_t$.

The competitive equilibrium of the IDF economy is a set of processes $\{d_t, c_t, h_t, k_{t+1}, \eta_t, \lambda_t, A_t\}$ satisfying (4.12), (4.56), (4.59), and (4.63)-(4.67), given the initial conditions A_0, d_{-1}, and k_0 and the exogenous process $\{\epsilon_t\}$.

We pick the same functional forms as in the EDF model. The fact that both the period utility function and the discount factor have a GHH structure implies that, as in all versions of the SOE RBC model considered thus far, the marginal rate of substitution between consumption and leisure depends only on hours worked and is independent of consumption. This yields the, by now familiar, equilibrium condition $h_t^{\omega-1} = A_t F_h(k_t, h_t)$.

The steady state of the IDF economy is the same as that of the EDF economy. To see this, note that in the steady state, (4.63) implies that $\beta(c, h)(1 + r^*) = 1$, which also features in the EDF model. Also, in the steady state, equation (4.67) yields an expression for the capital-labor ratio that is the same as in all versions of the SOE RBC model considered thus far. Finally, the fact that the labor supply schedule and the sequential budget constraint are identical in the EDF and IDF models, implies that h, c, and d are also equal across the two models. This shows that the IDF model delivers a steady-sate value of debt that is independent of initial conditions. Of course, the IDF model includes the variable η_t, which does not feature in the EDF model. The steady-state value of this variable is given by $-U(c, h)/r^*$.

Finally, we assign the same values to the structural parameters as in the EDF model.

4.10.5 The Model With No Stationarity Inducing Features (NSIF)

For comparison with the models studied thus far, we now consider a version of the small open economy RBC model featuring no stationarity inducing features. In this model (a) the discount factor is constant; (b) the interest rate at which domestic agents borrow from the rest of the world is constant (and equal to the subjective discount rate, $\beta(1+r^*)=1$); (c) agents face no frictions in adjusting the size of their asset portfolios; and (d) markets are incomplete, in the sense that domestic households have only access to a single risk-free international bond. Under this specification, the deterministic steady state of consumption depends on the assumed initial level of net foreign debt. Also, up to first order, the equilibrium dynamics contain a random walk component in variables such as consumption, the trade balance, and net external debt.

A competitive equilibrium in the nonstationary model is a set of processes $\{d_t, c_t, h_t, k_{t+1}, A_t\}$ satisfying (4.11), (4.12), (4.18), (4.56), (4.59), and the consumption Euler equation

$$U_c(c_t, h_t) = \beta(1+r^*)E_t U_c(c_{t+1}, h_{t+1}),$$

given d_{-1}, k_0, A_0, and the exogenous process $\{\epsilon_t\}$.

It is clear from the above consumption Euler equation that in the present model the marginal utility of consumption follows a random walk (recall that $\beta(1+r^*)=1$). This property is transmitted to consumption, debt, and the trade balance. Also, because the above Euler equation imposes no restriction in the deterministic steady state, the steady-state values of consumption, debt, and the trade balance are all indeterminate. The model does deliver unique deterministic steady-state values for k_t and h_t. We calibrate the parameters σ, r^*, ω, α, ϕ, δ, ρ, and $\widetilde{\eta}$ using the values displayed in tables 4.1.

4.10.6 The Perpetual-Youth Model (PY)

In this subsection, we present an additional way to induce stationarity in the small open economy RBC model. It is a discrete-time, stochastic, small-open-economy version of the perpetual-youth model due to Blanchard (1985). Cardia (1991) represents an early adoption of the perpetual-youth model in the context of a small open economy. Our model differs from Cardia's in that we assume a preference specification that allows for an exact aggregation of this model. Our strategy avoids the need to resort to linear approximations prior to aggregation.

The Basic Intuition

The basic intuition behind why the assumption of finite lives by itself helps to eliminate the unit root in the aggregate net foreign asset position can be seen from the following simple example. Consider an economy in which debt holdings of individual agents follow a pure random walk of the form $d_{s,t} = d_{s,t-1} + \mu_t$. Here, $d_{s,t}$ denotes the net debt position at the end of period t of an agent born in period s, and μ_t is an exogenous shock common to all agents and potentially serially correlated. This is exactly the equilibrium evolution of debt we obtained in the quadratic-preference, representative-agent economy of chapter 2, see equation (**??**). We now depart from the representative-agent assumption by introducing a constant and age-independent probability of death at the individual level. Specifically, assume that the population is constant over time and normalized to unity. Each period, individual agents face a probability $1-\theta \in (0,1)$ of dying. In addition, to keep the size of the population constant over time, we assume that $1-\theta$ agents are born each period. Assume that those agents who die leave their outstanding debts unpaid and that newborns inherit no debts. Adding the left- and right-hand sides of the law of motion for debt over all agents alive in period t—i.e., applying the operator $(1-\theta)\sum_{s=t}^{-\infty}\theta^{t-s}$ on both sides of the expression $d_{s,t} = d_{s,t-1} + \mu_t$—yields $d_t = \theta d_{t-1} + \mu_t$, where d_t denotes the aggregate debt position in period t. In performing the aggregation, recall that $d_{t,t-1} = 0$, because agents are born free of debts. Clearly, the resulting law of motion for the aggregate level of debt is mean reverting at the survival rate θ. The key difference with the representative agent model is that here each period a fraction $1-\theta$ of the stock of debt simply disappears.

In what follows, we embed this basic stationarity result into the small-open-economy real-business-cycle model.

Households

Each agent maximizes the utility function

$$-\frac{1}{2}E_0\sum_{t=0}^{\infty}(\beta\theta)^t(x_{s,t}-\overline{x})^2$$

with

$$x_{s,t}=c_{s,t}-\frac{h_{s,t}^{\omega}}{\omega}, \tag{4.68}$$

where $c_{s,t}$ and $h_{s,t}$ denote consumption and hours worked in period t by an agent born in period s. The parameter $\beta \in (0,1)$ represents the subjective discount factor, and $\overline{x}$ is a parameter denoting a satiation point. The symbol E_t denotes the conditional expectations operator over aggregate states. Following the preference specification used in all of the models studied in this chapter, we assume that agents derive utility from a quasi-difference between consumption and leisure. But we depart from the preference specifications used earlier in this chapter by assuming a quadratic period utility index. As will become clear shortly, this assumption is essential to achieve aggregation in the presence of aggregate uncertainty.

Financial markets are incomplete. Domestic consumers can borrow internationally by means of a bond paying a constant real interest rate. The debts of deceased domestic consumers are assumed to go unpaid. Foreign agents are assumed to lend to a large number of domestic consumers so that the fraction of unpaid loans due to death is deterministic. To compensate foreign lenders for these losses, domestic consumers pay a constant premium over the world interest rate. Specifically, the gross interest rate at which domestic consumers borrow internationally is $(1+r^*)/\theta$, where r^* denotes the world interest rate. Domestic agents can also lend internationally. The lending contract stipulates that should the domestic lender die, the foreign borrower is relieved of his debt obligations. Since foreing borrowers can perfectly diversify their loans across dometic agents, they pay a deterministic interest rate. To eliminate pure arbitrage opportunities, domestic consumers must lend at the rate $(1+r^*)/\theta$. It follows that the gross interest rate on the domestic consumer's asset position (whether this position is positive or negative) is given by $(1+r^*)/\theta$.

The budget constraint of a domestic consumer born in period $s \le t$ is

$$d_{s,t} = \left(\frac{1+r^*}{\theta}\right) d_{s,t-1} + c_{s,t} - \pi_t - w_t h_{s,t}, \tag{4.69}$$

where π_t and w_t denote, respectively, profits received from the ownership of stock shares and the real wage rate. To facilitate aggregation, we assume that agents do not trade shares and that the shares of the dead are passed to the newborn in an egalitarian fashion. Thus, share holdings are identical across agents. Agents are assumed to be subject to the following no-Ponzi-game constraint

$$\lim_{j\to\infty} E_t \left(\frac{\theta}{1+r^*}\right)^j d_{s,t+j} \le 0. \tag{4.70}$$

The first-order conditions associated with the agent's maximization problem are (4.68), (4.69), (4.70) holding with equality, and

$$-\left(x_{s,t} - \overline{x}\right) = \lambda_{s,t}, \tag{4.71}$$

$$h_{s,t}^{\omega-1} = w_t, \tag{4.72}$$

and

$$\lambda_{s,t} = \beta(1+r^*) E_t \lambda_{s,t+1}. \tag{4.73}$$

Note that $h_{s,t}$ is independent of s (i.e., it is independent of the agent's birth date). This means that we can drop the subscript s from $h_{s,t}$ and write

$$h_t^{\omega-1} = w_t. \tag{4.74}$$

Use equations (4.68) and (4.72) to eliminate $c_{s,t}$ from the sequential budget constraint (4.69). This yields

$$d_{s,t} = \left(\frac{1+r^*}{\theta}\right) d_{s,t-1} - \pi_t - \left(1 - \frac{1}{\omega}\right) w_t h_t + \overline{x} + (x_{s,t} - \overline{x}).$$

To facilitate notation, we introduce the auxiliary variable

$$z_t \equiv \pi_t + \left(1 - \frac{1}{\omega}\right) w_t h_t - \overline{x}, \tag{4.75}$$

which is the same for all generations s because both profits and hours worked are independent of the age of the cohort. Then the sequential budget constraint becomes

$$d_{s,t} = \left(\frac{1+r^*}{\theta}\right) d_{s,t-1} - z_t + (x_{s,t} - \overline{x}). \tag{4.76}$$

Now iterate this expression forward, apply the E_t operator, and use the transversality condition (i.e., equation (4.70) holding with equality), to obtain

$$\left(\frac{1+r^*}{\theta}\right) d_{s,t-1} = E_t \sum_{j=0}^{\infty} \left(\frac{\theta}{1+r^*}\right)^j \left[z_{t+j} - (x_{s,t+j} - \overline{x})\right].$$

Using equations (4.71) and (4.73) to replace $E_t x_{s,t+j}$ yields

$$\left(\frac{1+r^*}{\theta}\right) d_{s,t-1} = E_t \sum_{j=0}^{\infty} \left(\frac{\theta}{1+r^*}\right)^j z_{t+j} - \frac{\beta(1+r^*)^2}{\beta(1+r^*)^2 - \theta}(x_{s,t} - \overline{x}).$$

Solve for $x_{s,t}$ to obtain

$$x_{s,t} = \overline{x} + \frac{\beta(1+r^*)^2 - \theta}{\beta\theta(1+r^*)}(\widetilde{z}_t - d_{s,t-1}), \tag{4.77}$$

where

$$\widetilde{z}_t \equiv \frac{\theta}{1+r^*} E_t \sum_{j=0}^{\infty} \left(\frac{\theta}{1+r^*}\right)^j z_{t+j}$$

denotes the weighted average of current and future expected values of z_t. It can be expressed recursively as

$$\widetilde{z}_t = \frac{\theta}{1+r^*} z_t + \frac{\theta}{1+r^*} E_t \widetilde{z}_{t+1}. \tag{4.78}$$

We now aggregate individual variables by summing over generations born at time $s \leq t$. Notice that at time t there are alive $1-\theta$ people born in t, $(1-\theta)\theta$ people born in $t-1$, and, in general, $(1-\theta)\theta^s$ people born in period $t-s$. Let

$$x_t \equiv (1-\theta) \sum_{s=t}^{-\infty} \theta^{t-s} x_{s,t}$$

and

$$d_t \equiv (1-\theta) \sum_{s=t}^{-\infty} \theta^{t-s} d_{s,t}$$

denote the aggregate levels of $x_{s,t}$ and $d_{s,t}$, respectively. Now multiply (4.77) by $(1-\theta)\theta^{t-s}$ and then sum for $s = t$ to $s = -\infty$ to obtain the following expression for the aggregate version of equation (4.77):

$$x_t = \overline{x} + \frac{\beta(1+r^*)^2 - \theta}{\beta\theta(1+r^*)}(\widetilde{z}_t - \theta d_{t-1}). \tag{4.79}$$

In performing this step, keep in mind that $d_{t,t-1} = 0$. That is, consumers are born debt free.

Finally, aggregate the first-order condition (4.71) and the budget constraint(4.76) to obtain

$$-(x_t - \overline{x}) = \lambda_t \tag{4.80}$$

and

$$d_t = (1 + r^*)d_{t-1} - z_t + x_t - \overline{x}, \tag{4.81}$$

where

$$\lambda_t \equiv (1-\theta)\sum_{s=t}^{-\infty}\theta^{t-s}\lambda_{s,t}.$$

denotes the cross-sectional average of marginal utilities of consumption.

Firms Producing Consumption Goods

We assume the existence of competitive firms that hire capital and labor services to produce consumption goods. These firms maximize profits, which are given by

$$A_t F(k_t, h_t) - w_t h_t - u_t k_t,$$

where the function F and the productivity factor A_t are as in the EDEIR model. The first-order conditions associated with the firm's profit-maximization problem are

$$A_t F_k(k_t, h_t) = u_t \tag{4.82}$$

and

$$A_t F_h(k_t, h_t) = w_t. \tag{4.83}$$

We assume perfect competition in product and factor markets. Because F is homogeneous of degree one, firms producing consumption goods make zero profits.

Firms Producing Capital Goods

We assume the existence of firms that buy consumption goods to transform them into investment goods, rent out capital, and pay dividends, π_t. Formally, dividends in period t are given by

$$\pi_t = u_t k_t - i_t - \Phi(k_{t+1} - k_t). \tag{4.84}$$

The evolution of capital follows the law of motion given in (4.4), which we reproduce here for convenience

$$k_{t+1} = (1-\delta)k_t + i_t. \tag{4.85}$$

The optimization problem of the capital producing firm is dynamic. This is because investment goods take one period to become productive capital and because of the presence of adjustment costs. The firm must maximize some present discounted value of current and future expected profits. A problem that emerges at this point is what discount factor should the firm use. This issue does not have a clear answer for two reasons: first, the owners of the firm change over time. Recall that the shares of the dead are distributed in equal parts among the newborn. It follows that the firm cannot use as its discount factor the intertemporal marginal rate of substitution of a 'representative household.' For the representative household does not exist. Second, the firm operates in a financial environment characterized by incomplete asset markets. For this reason, it cannot use the price of state-contingent claims to discount future profits. For there is no market for such claims.

One must therefore introduce assumptions regarding the firm's discounting behavior. These assumptions will in general not be innocuous with respect to the dynamics of capital accumulation. With this in mind, we will assume that the firm uses the discount factor $\beta^j \lambda_{t+j}/\lambda_t$ to calculate the period-t value of one unit of consumption delivered in a particular state of period $t+j$. Note that this discount factor uses the average marginal utility of consumption of agents alive in period $t+j$ relative to the average marginal utility of consumption of agents alive in period t. Note that we use as the subjective discount factor the parameter β and not $\beta\theta$. This is because the number of shareholders is constant over time (and equal to unity), unlike the size of a cohort born at a particular date, which declines at the mortality rate $1-\theta$. The Lagrangian associated with the optimization problem of capital goods producers is then given by

$$\mathcal{L} = E_t \sum_{j=0}^{\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} \left[u_{t+j} k_{t+j} - k_{t+j+1} + (1-\delta) k_{t+j} - \Phi(k_{t+j+1} - k_{t+j}) \right].$$

The first-order condition with respect to k_{t+1} is

$$\lambda_t [1 + \Phi'(k_{t+1} - k_t)] = \beta E_t \lambda_{t+1} \left[u_{t+1} + 1 - \delta + \Phi'(k_{t+2} - k_{t+1}) \right]. \tag{4.86}$$

Equilibrium

Equations (4.74), (4.75), (4.78)-(4.86) form a system of eleven equations in eleven unknowns: x_t, λ_t, h_t, w_t, u_t, π_t, i_t, k_t, d_t, z_t, $\widetilde{z}_t$. Here, we reproduce the system of equilibrium conditions for convenience:

$$h_t^{\omega-1} = w_t,$$

$$z_t \equiv \pi_t + \left(1 - \frac{1}{\omega}\right) w_t h_t - \overline{x},$$

$$\widetilde{z}_t = \frac{\theta}{1+r^*} z_t + \frac{\theta}{1+r^*} E_t \widetilde{z}_{t+1},$$

$$x_t = \overline{x} + \frac{\beta(1+r^*)^2 - \theta}{\beta\theta(1+r^*)} (\widetilde{z}_t - \theta d_{t-1}),$$

$$-(x_t - \overline{x}) = \lambda_t,$$

$$d_t = (1+r^*)d_{t-1} - z_t + x_t - \overline{x},$$

$$A_t F_k(k_t, h_t) = u_t,$$

$$A_t F_h(k_t, h_t) = w_t,$$

$$\pi_t = u_t k_t - i_t - \Phi(k_{t+1} - k_t),$$

$$k_{t+1} = (1-\delta)k_t + i_t,$$

$$\lambda_t[1 + \Phi'(k_{t+1} - k_t)] = \beta E_t \lambda_{t+1}\left[u_{t+1} + 1 - \delta + \Phi'(k_{t+2} - k_{t+1})\right].$$

It is of interest to consider the special case in which $\beta(1+r^*) = 1$. In this case, the evolution of external debt is given by $d_t = \theta d_{t-1} + (1+r^*-\theta)/\theta \widetilde{z}_t - z_t$. This expression shows that the stock of debt does not follow a random walk as was the case in the representative-agent economy with quadratic preferences of chapter 2. In fact, the (autoregressive) coefficient on past external debt is $\theta \in (0,1)$. The mean reverting property of aggregate external debt obtains in spite of the fact that individual debt positions follow a random walk. The reason why the aggregate level of external debt is trend reverting in equilibrium is the fact that each period a fraction $1-\theta \in (0,1)$ of the agents die and are replaced by newborns holding no financial assets. As a result, on average, the current aggregate level of debt is only a fraction θ of the previous period's level of debt. This intuition also goes through when $\beta(1+r^*) \neq 1$, although in this case individual levels of debt display a trend in the deterministic equilibrium.

In the deterministic steady state, the aggregate level of debt is given by

$$d = \frac{\theta(1-\beta(1+r^*))}{(1+r^*-\theta)(\theta-\beta(1+r^*))} y$$

In the special case in which $\beta(1+r^*)$ equals unity, the steady-state aggregate stock of debt is nil.

This is because in this case agents, all of whom are born with no debts, wish to hold constant debt levels over time. In this case, the steady state both the aggregate and the individual levels of debt are zero. It can be shown that if $\beta(1+r^*)$ is less than unity but larger than θ, the steady-state level of debt must be positive.

We adopt the same functional forms for F and Φ as in the EDEIR model. We calibrate ω, α, ϕ, δ, ρ, β and $\widetilde{\eta}$ at the values displayed in tables 4.1. Consequently, the steady-state values of hours, capital, output, investment, consumption, and the trade balance are the same as in the EDEIR model. We set $\theta = 1 - 1/75$, which implies a life expectancy of 75 years. Finally, we calibrate r^* and $\overline{x}$ to ensure that in the steady state the trade-balance-to-output ration is 2 percent and the degree of relative risk aversion, given by $-x/(x-\overline{x})$, is 2. This calibration results in an interest rate of 3.7451 percent and a satiation point of 0.6334.

4.10.7 Quantitative Results

Table 4.4 displays a number of unconditional second moments of interest implied by the IDF, EDF, EDEIR, IDEIR, PAC, CAM, and PY models. The NSIF model is nonstationary up to first order, and therefore does not have well defined unconditional second moments. The second moments for all models other than the IDEIR and PY models are taken from Schmitt-Grohé and Uribe (2003). We compute the equilibrium dynamics by solving a log-linear approximation to the set of equilibrium conditions. The Matlab computer code used to compute the unconditional second moments and impulse response functions for all models presented in this section is available at `www.columbia.edu/~mu2166/closing.htm`.

Table 4.4 shows that regardless of how stationarity is induced, the model's predictions regarding second moments are virtually identical. One noticeable difference arises in the CAM model, the complete markets case, which, as might be expected, predicts less volatile consumption. The low volatility of consumption in the complete markets model introduces a difference between the

Table 4.4: Second Moments Across Models

	IDF	EDF	IDEIR	EDEIR	PAC	CAM	PY
Volatilities:							
std(y_t)	3.1	3.1	3.1	3.1	3.1	3.1	3.1
std(c_t)	2.3	2.3	2.5	2.7	2.7	1.9	2.5
std(i_t)	9.1	9.1	9	9	9	9.1	8.7
std(h_t)	2.1	2.1	2.1	2.1	2.1	2.1	2.1
std($\frac{tb_t}{y_t}$)	1.5	1.5	1.6	1.8	1.8	1.6	1.5
std($\frac{ca_t}{y_t}$)	1.5	1.5	1.4	1.5	1.5	3.1	1.3
Serial Correlations:							
corr(y_t, y_{t-1})	0.61	0.61	0.62	0.62	0.62	0.61	0.62
corr(c_t, c_{t-1})	0.7	0.7	0.76	0.78	0.78	0.61	0.74
corr(i_t, i_{t-1})	0.07	0.07	0.068	0.069	0.069	0.07	0.064
corr(h_t, h_{t-1})	0.61	0.61	0.62	0.62	0.62	0.61	0.62
corr($\frac{tb_t}{y_t}, \frac{tb_{t-1}}{y_{t-1}}$)	0.33	0.32	0.43	0.51	0.5	0.39	0.34
corr($\frac{ca_t}{y_t}, \frac{ca_{t-1}}{y_{t-1}}$)	0.3	0.3	0.31	0.32	0.32	-0.07	0.29
Correlations with Output:							
corr(c_t, y_t)	0.94	0.94	0.89	0.84	0.85	1	0.94
corr(i_t, y_t)	0.66	0.66	0.68	0.67	0.67	0.66	0.69
corr(h_t, y_t)	1	1	1	1	1	1	1
corr($\frac{tb_t}{y_t}, y_t$)	-0.012	-0.013	-0.036	-0.044	-0.043	0.13	-0.06
corr($\frac{ca_t}{y_t}, y_t$)	0.026	0.025	0.041	0.05	0.051	-0.49	0.04

Note. Standard deviations are measured in percent per year. IDF = Internal Discount Factor; EDF = External Discount Factor; IDEIR = Internal Debt-Elastic Interest Rate; EDEIR = External Debt-Elastic Interest Rate; PAC = Portfolio Adjustment Costs; CAM = Complete Asset Markets; PY = Perpetual Youth Model. Parts of the table are reproduced from Schmitt-Grohé and Uribe (2003).

predictions of this model and those of the IDF, EDF, EDEIR, IDEIR, PAC, and PY models: Because consumption is smoother in the CAM model, its role in determining the cyclicality of the trade balance is smaller. As a result, the CAM model predicts that the correlation between output and the trade balance is positive, whereas the models featuring incomplete asset markets all imply that this correlation is negative.

Figure 4.3 demonstrates that all of the models being compared imply virtually identical impulse response functions to a technology shock. Each panel shows the impulse response of a particular variable in the eight models. For all variables, the impulse response functions are so similar that to the naked eye the graph appears to show just a single line. Again, the only small and barely noticeable difference is given by the responses of consumption and the trade-balance-to-GDP ratio in the complete markets model. In response to a positive technology shock, consumption increases less when markets are complete than when markets are incomplete. This in turn, leads to a smaller decline in the trade balance in the period in which the technology shock occurs.

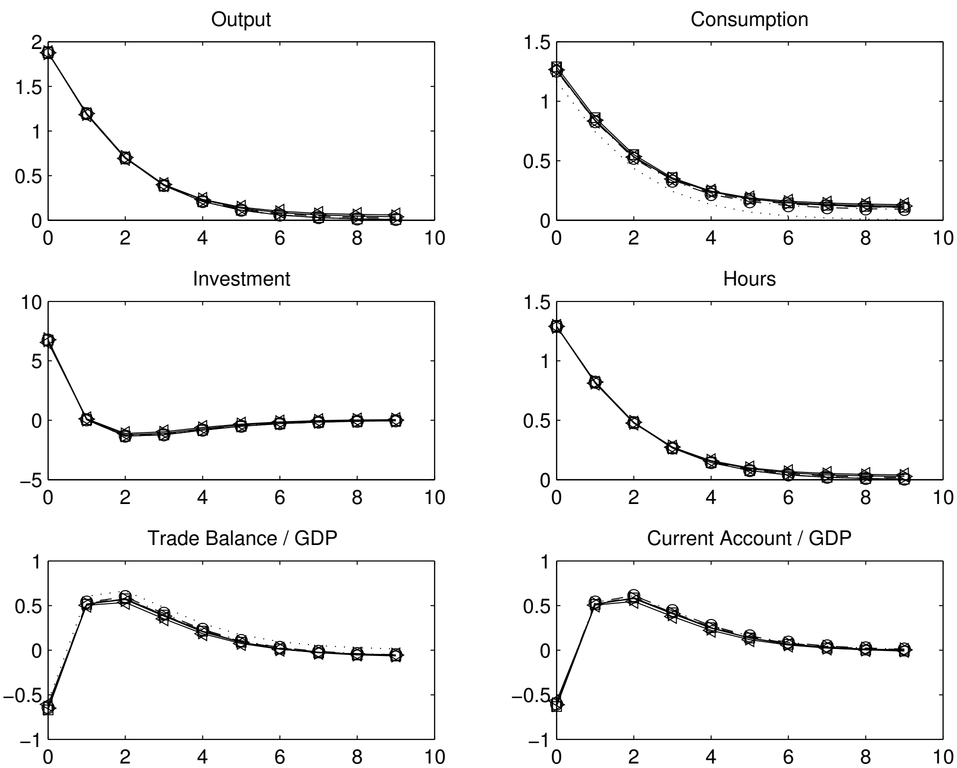
4.10.8 Inducing Stationarity Through Impatience and Global Solutions

Thus far, we have approximated the SOE-RBC model using a first-order local approximation around the deterministic steady state. Here, we approximate the equilibrium using a global method. We assume that both the subjective and pecuniary discount factors are constant. To induce stationarity, we assume that households are slightly more impatient than indicated by the pecuniary discount factor, that is, we assume that

$$\beta(1 + r^*) < 1.$$

In all other respects, the model is exactly as the baseline SOE model of section 4.1. The above restriction eliminates the random walk in debt and consumption. To understand why, assume first that the economy is deterministic. In such an environment, the condition $\beta(1 + r^*) < 1$ induces

Figure 4.3: Impulse Response to a Unit Technology Shock Across Models



Note. Solid line, IDF model; squares, EDF; dashed line, EDEIR model; dash-dotted line, PAC model; dotted line, CAM model; circles, NSIF model; right triangle, IDEIR model; left triangle, PY.

households to accumulate debt in such a way that consumption approaches asymptotically its lowest possible level, given by the disutility of labor, h^ω/ω. Thus, debt converges to a well-defined value given by the present discounted value of the stream of output, $AF(k,h)$, net of capital depreciation, δk, and consumption, h^ω/ω, that is, $d \to [AF(k,h) - \delta k - h^\omega/\omega]/r^*$, where all variables are evaluated at their asymptotic values. Of course, at the limit the household is infinitely unhappy. Thus, when uncertainty is introduced, in the form of stochastic variations in the technology shock A, precautionary savings create a well-defined debt distribution to the left of its asymptotic deterministic level. Since this precautionary savings motive cannot be captured by a first-order approximation, it follows that a higher-order approximation, like the global approximation pursued here, is needed to obtain a stationary solution.

The solution algorithm is based on value function iterations over a discretized state space. Specifically, we write the model as the solution to the Bellman equation

$$v(d,k,A) = \max_{\{d',k'\}} \left\{ U\left(AF(k,h) + (1-\delta)k - k' - \Phi(k'-k) + d' - (1+r^*)d - \frac{h^\omega}{\omega} \right) + \beta E[v(d',k',A')|A] \right\}$$

subject to

$$d' \leq \overline{d},$$

where variables without a subscript or a superscript are dated in period t and primed variables are dated in $t+1$. The debt constraint places a limit to the level of net external debt, defined by the parameter $\overline{d}$. When $\overline{d}$ is set to a large number so that the debt constraint is never binding the debt constraint serves only as a no-Ponzi-game restriction. But, as it will become clear shortly, a more stringent debt limit is needed for the predictions of the model to be in line with the data. The online materials for this chapter provide the transition probability matrix of the exogenous shock A (file tpm.mat) and the MATLAB script to compute the value function and the associated policy functions (file vfi.m).

Table 4.5: Calibration of the Discount Factor and the Debt Limit

Description	β	$\beta(1+r^*)$	$\bar{d}$	$E(tb/y)$	σ_i
Chosen Calibration	0.954	0.9922	1	2.9	8.7
Natural Debt Limit	0.954	0.9922	9.95	25.8	21.40
High Patience	0.96	0.9984	1	2.6	9.9
Data			2.0	9.8	

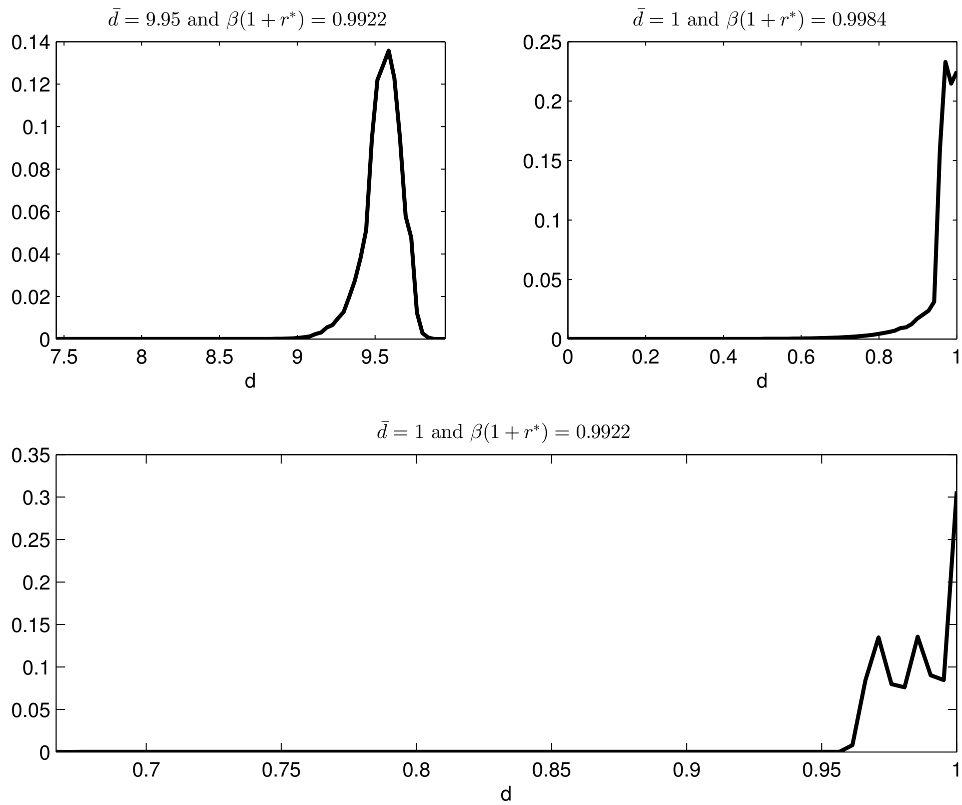
Note. $r^* = 0.04$.

We use the same functional forms for preferences, technologies and adjustment costs as in previous sections and calibrate all parameters, except β and $\overline{d}$, as shown in table 4.1. We discretize the state space with 9 equally spaced points for $\ln A$ from -0.04495 to 0.04495, 70 equally spaced points for d, and 30 equally spaced points for k. The ranges assigned to the two endogenous states, d and k, depend on the values assigned to β and $\overline{d}$, as explained below.

Setting $\beta(1 + r^*)$ too close to unity results in a near random walk distribution of debt in equilibrium, which is difficult to approximate. We therefore set $\beta(1 + r^*)$ equal to 0.9922, which, recalling that $r^* = 0.04$, requires setting β equal to 0.954. This value of β introduces a significant difference of 82 basis points between the subjective and the pecuniary discount rates.

Consider first the case in which the debt limit $\bar{d}$ is set to a large number so that the borrowing constraint is never binding in equilibrium. Specifically we set the range of d to be $[7.45, 9.95]$. The grid for k is [2.8,3.8]. The top-left panel of figure 4.4 displays the equilibrium probability distribution of d. The probability of debt being equal to its upper limit of 9.95 is zero, which implies that the debt limit is never binding in equilibrium. However, the implied average level of external debt, equal to 9.6, turns out to be excessive. Table 4.5 explains why. It shows that in order to support such a high level of debt the economy must generate a trade balance surplus of 25.8 percent of GDP, more than 10 times larger than the one observed in Canada, the economy to which the model is calibrated. Under this calibration, the model is at odds with the data along

Figure 4.4: Probability Distribution of Debt Under Impatience and a Global Approximation



Note. $r^* = 0.04$.

other dimensions as well. For instance, the table shows that the predicted volatility of investment is 21.4 percent, more than twice what it is in the data. The intuition why investment is so volatile is that at high levels of external debt, the intertemporal marginal rate of consumption substitution $U'(c' - h'^{\omega}/\omega)/U'(c - h^{\omega}/\omega)$ becomes highly volatile. Recall that this variable represents the kernel households use to discount future returns to capital (see equations (4.8) and (4.10)). In turn, the reason why the intertemporal marginal rate of consumption substitution is highly volatile is that consumption is close to its minimum possible level, given by the disutility of labor h^{ω}/ω. When $c - h^{\omega}/\omega$ is close to zero, the marginal utility of consumption becomes highly sensitive to changes in c or h.

It follows that a tighter limit on external debt is in order. Accordingly, we set $\overline{d}$ to 1. As shown in table 4.5, this value is low enough to induce an average trade-balance-to-output ratio close to the observed value of 2 percent. The table also shows that with the tighter debt limit, the model improves significantly in explaining the observed volatility of investment (9.8 in the data versus 8.7 in the model). Indeed, the model does a pretty good job at explaining other second moments of interest as well. Table 4.6 displays the standard deviation, serial correlation, and correlation with output of output, consumption, investment, hours, the trade-balance-to-output ratio and the current-account-to-output ratio. For comparison, the table reproduces form table 4.2 the corresponding empirical moments and those implied by the EDEIR model.

The main message of the table is that the model with impatient agents solved using global methods performs as well as the EDEIR model solved with local approximation methods except for the serial correlation of the trade-balance-to-output ratio, which is better captured by the log-linearized EDEIR model. A similar result emerges from figure 4.5, displaying impulse responses to a positive productivity shock. Comparing this figure with figure 4.3, shows that the response of the model with impatient households obtained via a global approximation is similar to that obtained under local approximations and alternative stationarity inducing mechanisms.

Figure 4.5: Impulse Responses to a Positive Productivity Shock Under a Global Approximation

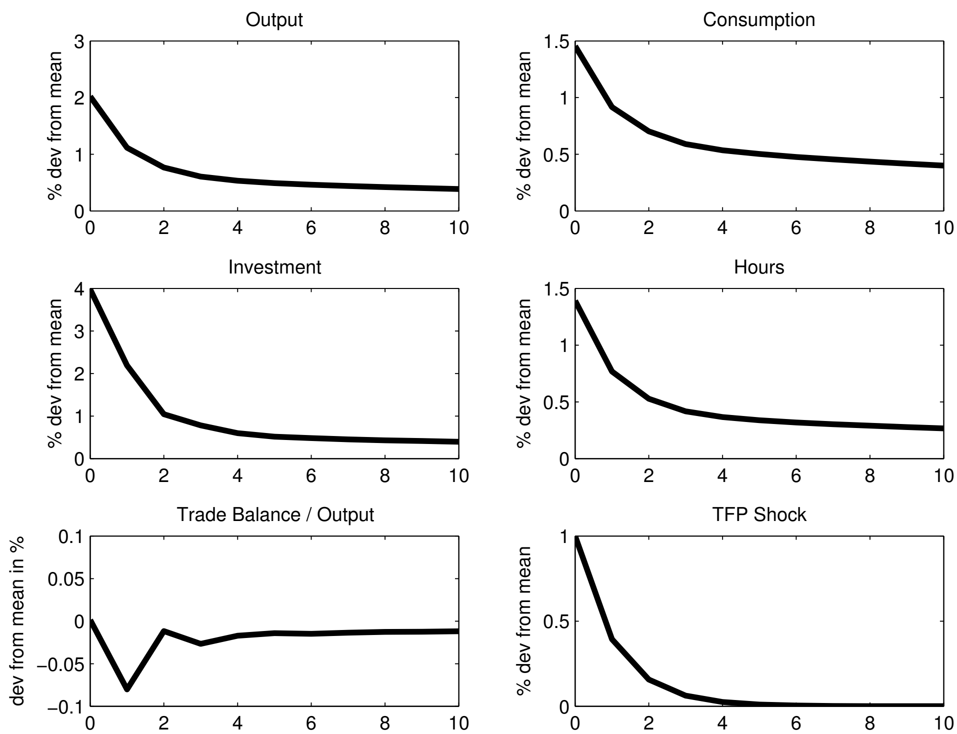


Table 4.6: Model Predictions Under A Global Approximation

Variable	Data			Global Solution			EDEIR		
	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,GDP_t}	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,GDP_t}	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,GDP_t}
y	2.81	0.62	1	3.56	0.68	1	3.08	0.62	1
c	2.46	0.70	0.59	2.96	0.77	0.98	2.71	0.78	0.84
i	9.82	0.31	0.64	8.70	0.09	0.69	9.04	0.07	0.67
h	2.02	0.54	0.80	2.44	0.68	1	2.12	0.62	1
$\frac{tb}{y}$	1.87	0.66	-0.13	1.23	-0.46	-0.04	1.78	0.51	-0.04
$\frac{ca}{y}$				1.21	-0.47	0.05	1.45	0.32	0.05

Note. In the global solution $\beta(1+r^*)$ takes the value 0.9922 and $\bar{d}$ is equal to 1. Second moments are computed using the program vfi.m available with the online materials for this chapter. For second moments predicted by the EDEIR model and for empirical second moments, see the note to table 4.2.

Finally, we note that the predictions implied by the global approximation are robust to a wide range of values of β. Table 4.5 shows that both the trade-balance-to-output ratio and the volatility of investment are little changed if β is raised to 0.96, a value quite close to $1/(1+r^*) = 0.9615$. This value might be empirically appealing because it implies a lower frequency of events in which the debt limit binds, 22 percent with $\beta = 0.96$ versus 31 percent at the baseline value of 0.954 (compare the top-right and the bottom panels of figure 4.4).

4.11 Appendix: First-Order Accurate Approximations to Dynamic General Equilibrium Models

In this appendix, we solve the system

$$f_{y'} E_t \widehat{y}_{t+1} + f_y \widehat{y}_t + f_{x'} E_t \widehat{x}_{t+1} + f_x \widehat{x}_t \tag{4.41}$$

reproduced from section 4.6. The matrices $f_{y'}$, f_y, $f_{x'}$, and f_x are assumed to be known. Letting $A = [f_{x'} \quad f_{y'}]$ and $B = -[f_x \quad f_y]$, we can rewrite the system as

$$A \begin{bmatrix} E_t \widehat{x}_{t+1} \\ E_t \widehat{y}_{t+1} \end{bmatrix} = B \begin{bmatrix} \widehat{x}_t \\ \widehat{y}_t \end{bmatrix}.$$

Define the vector $\widehat{w}_t$ containing all control and state variables of the system. Formally

$$\widehat{w}_{t+j} \equiv E_t \begin{bmatrix} \widehat{x}_{t+j} \\ \widehat{y}_{t+j} \end{bmatrix}$$

for $j \geq 0$. Note that this definition implies that

$$\widehat{w}_t \equiv \begin{bmatrix} \widehat{x}_t \\ \widehat{y}_t \end{bmatrix}.$$

We can then write the linear system as

$$A \widehat{w}_{t+1} = B \widehat{w}_t.$$

In accordance with (4.40), we seek solutions in which

$$\lim_{j\to\infty} \widehat{w}_{t+j} = 0. \tag{4.87}$$

This requirement means that at every point in time the vector w_t is expected to converge to its non-stochastic steady state, $w \equiv [x' \quad y']'$.

The remainder of this section is based on Klein (2000) (see also Sims, 1996). Consider the generalized Schur decomposition of A and B:

$$qAz = a$$

and

$$qBz = b,$$

where a and b are upper triangular matrices and q and z are orthonormal matrices. Recall that a matrix a is said to be upper triangular if elements in row i and column j, denoted $a(i,j)$ are 0 for $i > j$. A matrix z is orthonormal if $z'z = zz' = I$.

Define

$$s_t \equiv z'\widehat{w}_t.$$

Then we have that

$$as_{t+1} = bs_t.$$

The ratio $b(i,i)/a(i,i)$ is known as the generalized eigenvalue of the matrices A and B. Assume, without loss of generality, that the ratios $|b(i,i)/a(i,i)|$ are increasing in i. Now partition a, b, z,

$\widehat{w}_t$, and s_t as

$$a = \begin{bmatrix} a_{11} & a_{12} \\ \emptyset & a_{22} \end{bmatrix}, \quad b = \begin{bmatrix} b_{11} & b_{12} \\ \emptyset & b_{22} \end{bmatrix}; \quad z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}; \quad \widehat{w}_t = \begin{bmatrix} \widehat{w}_t^1 \\ \widehat{w}_t^2 \end{bmatrix}, \quad s_t = \begin{bmatrix} s_t^1 \\ s_t^2 \end{bmatrix},$$

where a_{11} and b_{11} are square matrices whose diagonals generate the generalized eigenvalues of (A, B) with absolute values less than one, and a_{22} and b_{22} are square matrices whose diagonals generate the generalized eigenvalues of (A, B) with absolute values greater than one. Then we have that

$$a_{22} s_{t+1}^2 = b_{22} s_t^2.$$

The partition of the matrix B guarantees that all diagonal elements of b_{22} are nonzero. In addition, recalling that a triangular matrix is invertible if the elements along its main diagonal are nonzero, it follows, that b_{22} is invertible. So we can write

$$b_{22}^{-1} a_{22} s_{t+1}^2 = s_t^2.$$

By construction, the eigenvalues of $b_{22}^{-1} a_{22}$ are all less than unity in modulus. To arrive at this conclusion, we use three properties of upper triangular matrices: (a) the inverse of a nonsingular upper triangular matrix is upper triangular; (b) the product of two upper triangular matrices is upper triangular; and (c) the eigenvalues of an upper triangular matrix are the elements of its main diagonal. It follows that the only nonexplosive solution to the above difference equation is

$$s_t^2 =$$

for all t. This result, and the definition of s_t^2 imply that

$$z'_{12}\widehat{w}_t^1 + z'_{22}\widehat{w}_t^2 = 0.$$

Solving this expression for $\widehat{w}_t^2$ yields

$$\widehat{w}_t^2 = G\widehat{w}_t^1, \tag{4.88}$$

where

$$G \equiv -{z'_{22}}^{-1} z'_{12}. \tag{4.89}$$

The invertibility of z'_{22} follows from the fact that, being orthonormal, z' itself is invertible. The condition $s_t^2 = 0$ for all t also implies that

$$a_{11}s_{t+1}^1 = b_{11}s_t^1.$$

The criteria used to partition A and B guarantee that the diagonal elements of the upper triangular matrix a_{11} are nonzero. Therefore, a_{11} is invertible, which allows us to write

$$s_{t+1}^1 = a_{11}^{-1}b_{11}s_t^1. \tag{4.90}$$

Now express s_t^1 as a linear transformation of $\widehat{w}_t^1$ as follows:

$$\begin{aligned} s_t^1 &= z'_{11}\widehat{w}_t^1 + z'_{21}\widehat{w}_t^2 \\ &= (z'_{11} + z'_{21}G)\widehat{w}_t^1 \\ &= (z'_{11} - z'_{21}{z'_{22}}^{-1}z'_{12})\widehat{w}_t^1 \\ &= z_{11}^{-1}\widehat{w}_t^1 \end{aligned}$$

The second and third equalities make use of equation (4.88) and identity (4.89), respectively. The last equality follows from the fact that z is orthonormal.[4]

Combining this expression with (4.90) yields

$$\widehat{w}^1_{t+1} = H\widehat{w}^1_t,$$

where

$$H \equiv z_{11}a_{11}^{-1}b_{11}z_{11}^{-1}.$$

Finally, note that all eigenvalues of H are inside the unit circle. To see this note that the eigenvalues of $z_{11}a_{11}^{-1}b_{11}z_{11}^{-1}$ must be same as the eigenvalues of $a_{11}^{-1}b_{11}$. In turn $a_{11}^{-1}b_{11}$ is upper triangular with diagonal elements less than one in modulus.

4.12 Appendix: Local Existence and Uniqueness of Equilibrium

The analysis thus far has not delivered the matrices h_x and g_x that define the first-order accurate solution of the DSGE model. In this section, we accomplish this task and derive conditions under which the equilibrium dynamics are locally unique.

[4]To see this, let $k \equiv z'_{11} - z'_{21}{z'_{22}}^{-1}z'_{12}$. We with to show that $k = z_{11}^{-1}$. Note that the orthonormality of z implies that

$$I = z'z = \begin{bmatrix} z'_{11}z_{11} + z'_{21}z_{21} & z'_{11}z_{12} + z'_{21}z_{22} \\ z'_{12}z_{11} + z'_{22}z_{21} & z'_{12}z_{12} + z'_{22}z_{22} \end{bmatrix}.$$

Use element $(2,1)$ of $z'z$ to get $z'_{12}z_{11} = -z'_{22}z_{21}$. Pre-multiply by ${z'_{22}}^{-1}$ and post multiply by z_{11}^{-1} to get ${z'_{22}}^{-1}z'_{12} = -z_{21}{z_{11}}^{-1}$. Use this expression to eliminate ${z'_{22}}^{-1}z'_{12}$ from the definition of k to obtain $k = [z'_{11} + z'_{21}z_{21}{z_{11}}^{-1}]$. Now use element $(1,1)$ of $z'z$ to write $z'_{21}z_{21} = I - z'_{11}z_{11}$. Using this equation to eliminate $z'_{21}z_{21}$ from the expression in square brackets, we get $k = [z'_{11} + (I - z'_{11}z_{11}){z_{11}}^{-1}]$, which is simply z_{11}^{-1}. Finally, note that the invertibility of z_{11} follows from the invertibility of z.

4.12.1 Local Uniqueness of Equilibrium

Suppose that the number of generalized eigenvalues of the matrices A and B with absolute value less than unity is exactly equal to the number of states, n_x. That is, suppose that a_{11} and b_{11} are of size $n_x \times n_x$. In this case, the matrix H is also of size $n_x \times n_x$, and the matrix G is of size $n_y \times n_x$. Moreover, since $\widehat{w}_t^1$ must be comformable with H, we have that $\widehat{w}_t^1$ is given by the first n_x elements of $\widehat{w}_t$, which exactly coincide with $\widehat{x}_t$. In turn, this implies that $\widehat{w}_t^2$ must equal $\widehat{y}_t$. Defining

$$h_x \equiv H$$

and

$$g_x \equiv G,$$

we can then write

$$\widehat{x}_{t+1} = h_x \widehat{x}_t$$

and

$$\widehat{y}_t = g_x \widehat{x}_t,$$

which is the solution we were looking for. Notice that because $\widehat{x}_t$ is predetermined in period t, we have that $\widehat{y}_t$ and $\widehat{x}_{t+1}$ are uniquely determined in period t. The evolution of the linearized system is then unique and given by

$$y_t - y = g_x(x_t - y)$$

$$x_{t+1} - x = h_x(x_t - x) + \eta\epsilon_{t+1},$$

where we have set σ at the desired value of 1.

Summarizing, the condition for local uniqueness of the equilibrium is that the number of generalized eigenvalues of the matrices A and B is exactly equal to the number of states, n_x.

4.12.2 No Local Existence of Equilibrium

Now suppose that the number of generalized eigenvalues of the matrices A and B with absolute value less than one is smaller than the number of state variables, n_x. Specifically, suppose that a_11 and b_{11} are of size $(n_x - m) \times (n_x - m)$, with $0 < m \leq n_x$. In this case, the matrix H is of order $(n_x - m) \times (n_x - m)$ and the matrix G is of order $(n_y + m) \times (n_x - m)$. Moreover, the vectors $\widehat{w}_t^1$ and $\widehat{w}_t^2$ no longer coincide with $\widehat{x}_t$ and $\widehat{y}_t$, respectively. Instead, $\widehat{w}_t^1$ and $\widehat{w}_t^2$ take the form

$$\widehat{w}_t^1 = \widehat{x}_t^a$$

$$\widehat{w}_t^2 = \begin{bmatrix} \widehat{x}_t^b \\ \widehat{y}_t \end{bmatrix},$$

where $\widehat{x}_t^a$ and $\widehat{x}_t^b$ are vectors of lengths $n_x - m$ and m, respectively, and satisfy

$$\widehat{x}_t = \begin{bmatrix} \widehat{x}_t^a \\ \widehat{x}_t^b \end{bmatrix},$$

The law of motion of $\widehat{x}_t$ and $\widehat{y}_t$ is then of the form

$$\widehat{x}_{t+1}^a = H\widehat{x}_t^a$$

and

$$\begin{bmatrix} \widehat{x}_t^b \\ \widehat{y}_t \end{bmatrix} = G\widehat{x}_t^a$$

This expression states that $\widehat{x}_t^b$ is determined by $\widehat{x}_t^a$. But this is impossible, because $\widehat{x}_t^a$ and $\widehat{x}_t^b$ are predetermined independently of each other. We therefore say that locally there exists no equilibrium.

Summarizing, no local equilibrium exists if the number of generalized eigenvalues of the matrices A and B with absolute values less than one is smaller than the number of state variables, n_x.

4.12.3 Local Indeterminacy of Equilibrium

Finally, suppose that the number of generalized eigenvalues of the matrices A and B with absolute value less than one is larger than the number of state variables, n_x. Specifically, suppose that a_11 and b_{11} are of size $(n_x + m) \times (n_x + m)$, with $0 < m \leq n_y$. In this case, the matrix H is of order $(n_x + m) \times (n_x + m)$ and the matrix G is of order $(n_y - m) \times (n_x + m)$. The vectors $\widehat{w}_t^1$ and $\widehat{w}_t^2$ take the form

$$\widehat{w}_t^1 = \begin{bmatrix} \widehat{x}_t \\ \widehat{y}_t^a \end{bmatrix},$$

$$\widehat{w}_t^2 = \widehat{y}_t^b$$

where $\widehat{y}_t^a$ and $\widehat{y}_t^b$ are vectors of lengths m and $n^y - m$, respectively, and satisfy

$$\widehat{y}_t = \begin{bmatrix} \widehat{y}_t^a \\ \widehat{y}_t^b \end{bmatrix},$$

The law of motion of $\widehat{x}_t$ and $\widehat{y}_t$ is then of the form

$$\begin{bmatrix} \widehat{x}_{t+1} \\ \widehat{y}_{t+1}^a \end{bmatrix} = H \begin{bmatrix} \widehat{x}_t \\ \widehat{y}_t^a \end{bmatrix}$$

and

$$\widehat{y}_t^b = G \begin{bmatrix} \widehat{x}_t \\ \widehat{y}_t^a \end{bmatrix}$$

These expressions state that one can freely pick $\widehat{y}_t^a$ in period t. Since $\widehat{y}_t^a$ is not predetermined, the equilibrium is indeterminate. In this case, we say that the indeterminacy is of dimension m. The evolution of the system can then be written as

$$\begin{bmatrix} x_{t+1} - x \\ y_{t+1}^a - y^{ass} \end{bmatrix} = H \begin{bmatrix} x_t - x \\ y_t^a - y^{ass} \end{bmatrix} + \begin{bmatrix} \eta & \emptyset \\ \nu_\epsilon & \nu_\mu \end{bmatrix} \begin{bmatrix} \epsilon_{t+1} \\ \mu_{t+1} \end{bmatrix}$$

and

$$y_t^b - y^{bss} = G \begin{bmatrix} x_t - x \\ y_t^a - y^{ass} \end{bmatrix},$$

where the matrices ν_ϵ and ν_μ allow for nonfundamental uncertainty, and μ_t is an i.i.d. innovation with mean $\emptyset$ and variance covariance matrix equal to the identity matrix.

Summarizing, the equilibrium displays local indeterminacy of dimension m if the number of generalized eigenvalues of the matrices A and B with absolute values less than one exceeds the number of state variables, n_x, by $0 < m \leq n_y$.

4.13 Appendix: Second Moments

Start with the equilibrium law of motion of the deviation of the state vector with respect to its steady-state value, which is given by

$$\widehat{x}_{t+1} = h_x \widehat{x}_t + \sigma \eta \epsilon_{t+1}, \tag{4.91}$$

Covariance Matrix of x_t

Let

$$\Sigma_x \equiv E\widehat{x}_t\widehat{x}_t'$$

denote the unconditional variance/covariance matrix of $\widehat{x}_t$ and let

$$\Sigma_\epsilon \equiv \sigma^2\eta\eta'.$$

Then we have that

$$\Sigma_x = h_x\Sigma_x h_x' + \Sigma_\epsilon.$$

We will describe two numerical methods to compute Σ_x.

Method 1

One way to obtain Σ_x is to make use of the following useful result. Let A, B, and C be matrices whose dimensions are such that the product ABC exists. Then

$$\text{vec}(ABC) = (C' \otimes A) \cdot \text{vec}(B),$$

where the vec operator transforms a matrix into a vector by stacking its columns, and the symbol $\otimes$ denotes the Kronecker product. Thus if the vec operator is applied to both sides of

$$\Sigma_x = h_x\Sigma_x h_x' + \Sigma_\epsilon,$$

the result is

$$\begin{aligned} \text{vec}(\Sigma_x) &= \text{vec}(h_x \Sigma_x h_x') + \text{vec}(\Sigma_\epsilon) \\ &= \mathcal{F}\ \text{vec}(\Sigma_x) + \text{vec}(\Sigma_\epsilon), \end{aligned}$$

where

$$\mathcal{F} = h_x \otimes h_x.$$

Solving the above expression for $\text{vec}(\Sigma_x)$ we obtain

$$\text{vec}(\Sigma_x) = (I - \mathcal{F})^{-1}\text{vec}(\Sigma_\epsilon)$$

provided that the inverse of $(I - \mathcal{F})$ exists. The eigenvalues of $\mathcal{F}$ are products of the eigenvalues of the matrix h_x. Because all eigenvalues of the matrix h_x have by construction modulus less than one, it follows that all eigenvalues of $\mathcal{F}$ are less than one in modulus. This implies that $(I - \mathcal{F})$ is nonsingular and we can indeed solve for Σ_x. One possible drawback of this method is that one has to invert a matrix that has dimension $n_x^2 \times n_x^2$.

Method 2

The following iterative procedure, called doubling algorithm, may be faster than the one described above in cases in which the number of state variables (n_x) is large.

$$\Sigma_{x,t+1} = h_{x,t}\Sigma_{x,t}h_{x,t}' + \Sigma_{\epsilon,t}$$

$$h_{x,t+1} = h_{x,t}h_{x,t}$$

$$\Sigma_{\epsilon,t+1} = h_{x,t}\Sigma_{\epsilon,t}h_{x,t}' + \Sigma_{\epsilon,t}$$

$$\Sigma_{x,0} = I$$

$$h_{x,0} = h_x$$

$$\Sigma_{\epsilon,0} = \Sigma_\epsilon$$

Other second moments

Once the covariance matrix of the state vector, x_t has been computed, it is easy to find other second moments of interest. Consider for instance the covariance matrix $E\widehat{x}_t\widehat{x}'_{t-j}$ for $j > 0$. Let $\mu_t = \sigma\eta\epsilon_t$.

$$\begin{aligned} E\widehat{x}_t\widehat{x}'_{t-j} &= E[h_x^j\widehat{x}_{t-j} + \sum_{k=0}^{j-1} h_x^k\mu_{t-k}]\widehat{x}'_{t-j} \\ &= h_x^j E\widehat{x}_{t-j}\widehat{x}'_{t-j} \\ &= h_x^j\Sigma_x \end{aligned}$$

Similarly, consider the variance covariance matrix of linear combinations of the state vector x_t. For instance, the co-state, or control vector y_t is given by $y_t = y + g_x(x_t - x)$, which we can write as: $\widehat{y}_t = g_x\widehat{x}_t$. Then

$$\begin{aligned} E\widehat{y}_t\widehat{y}'_t &= Eg_x\widehat{x}_t\widehat{x}'_t g'_x \\ &= g_x[E\widehat{x}_t\widehat{x}'_t]g'_x \\ &= g_x\Sigma_x g'_x \end{aligned}$$

and, more generally,

$$\begin{aligned} E\widehat{y}_t\widehat{y}'_{t-j} &= g_x[E\widehat{x}_t\widehat{x}'_{t-j}]g'_x \\ &= g_x h_x^j\Sigma_x g'_x, \end{aligned}$$

for $j \geq 0$.

4.14 Appendix: Impulse Response Functions

The impulse response to a variable, say z_t in period $t+j$ to an impulse in period t is defined as:

$$IR(z_{t+j}) \equiv E_t z_{t+j} - E_{t-1} z_{t+j}$$

The impulse response function traces the expected behavior of the system from period t on given information available in period t, relative to what was expected at time $t-1$. Using the law of motion $E_t \widehat{x}_{t+1} = h_x \widehat{x}_t$ for the state vector, letting x denote the innovation to the state vector in period 0, that is, $x = \eta\sigma\epsilon_0$, and applying the law of iterated expectations we get that the impulse response of the state vector in period t is given by

$$IR(\widehat{x}_t) \equiv E_0 \widehat{x}_t - E_{-1} \widehat{x}_t = h_x^t [x_0 - E_{-1} x_0] = h_x^t [\eta\sigma\epsilon_0] = h_x^t x; \quad t \geq 0.$$

The response of the vector of controls $\widehat{y}_t$ is given by

$$IR(\widehat{y}_t) = g_x h_x^t x.$$

4.15 Appendix: Matlab Code For Linear Perturbation Methods

The program `gx_hx.m` computes the matrices g_x and h_x using the Schur decomposition method. The program `mom.m` computes second moments. The program `ir.m` computes impulse response functions.

4.16 Exercises

Exercise 4.1 (Variation of the Portfolio Adjustment Cost Model) *This exercise aims at establishing whether formulating portfolio adjustment costs as a function of the deviation of the household's debt position from an exogenous reference point,* $d_t - \bar{d}$, *or as a function of the change in its debt position,* $d_t - d_{t-1}$, *has consequences for the stationarity of the model.*

Consider a small open economy populated by a large number of infinitely lived households with preferences described by the utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t,$$

where $\beta \in (0, 1)$ *denotes the subjective discount factor and* c_t *denotes consumption in period* t. *Each period, households receive an exogenous and stochastic endowment,* y_t, *and can borrow (or lend to) international financial markets at the gross interest rate* $1 + r$. *Let* d_t *denote the stock of foreign debt held by households at the end of period* t. *Households are subject to a portfolio adjustment cost of the form* $\frac{\phi}{2}(d_t - d_{t-1})^2$, *where* ϕ *is a positive constant. Assume that* $\beta(1 + r) = 1$.

1. *State the household's period by period budget constraint.*

2. *State the household's utility maximization problem*

3. *Write the Lagrangian of the household's problem*

4. *Define a competitive equilibrium of this economy.*

5. *Suppose the endowment is non-stochastic and constant,* $y_t = y$ *for all* t. *Characterize the deterministic steady state. Does it exist? Is it unique?*

6. *Consider now a temporary endowment shock. Suppose* $y_0 > y$ *and* $y_t = y$ *for all* $t > 0$ *deterministically. Suppose that prior to period 0 the economy was in a deterministic steady*

state with $d_{-1} = d^*$. Is the economy stationary, that is, is d_t expected to return d^*? Provide intuition.

Exercise 4.2 (Variation of the EDF Model) *This exercise analyzes the local stability of the equilibrium of the SOE-EDF model when the household's subjective discount factor is assumed to be increasing in aggregate consumption, $\theta'(c_t) > 0$, as opposed to decreasing, as is assumed in the baseline specification presented in section 4.10.3.*

Consider a small open economy populated by infinitely-lived agents. Let c_t denote consumption in period t. Assume that the discount factor, denoted β_t, evolves over time according to $\beta_{t+1} = \theta(c_t)\beta_t$. Assume that the function θ is positive and bounded above by unity. Agents have access to international financial markets where they can borrow or lend at the interest rate $r > 0$. Agents choose consumption and external debt, d_t so as to maximize lifetime utility given by $\sum_{t=0}^{\infty} \beta_t U(c_t)$, where $U(.)$ is an increasing and strictly concave function. Agents are endowed with $y > 0$ units of goods each period. Agents enter period 0 with a stock d_{-1} of net foreign debt. Assume that $\beta_0 = 1$. Assume that households are subject to some borrowing constraint that prevents them from engaging in Ponzi schemes. Assume that agents fail to internalize that their consumption choices affect their discount factor.

1. *Characterize the competitive equilibrium of this economy.*

2. *Characterize the steady state of this economy. Consider the following two cases: (A.) θ is strictly increasing in c and (B.) θ is strictly decreasing in c. What properties does the function $\theta(.)$ need to have in each case to ensure existence of a steady state. What properties does the function θ need to have in each case to ensure that the steady state is unique. Provide an intuitive explanation for your results by comparing them to those you would obtain in an economy in which $\theta(.)$ is independent of c_t. Which case, (A.) or (B.) is more plausible to you and why?*

3. *Characterize the local stability of the economy in a small neighborhood around the steady state. Specifically, suppose that d_{-1} is not equal to the steady state, under what conditions (on the function θ) does there exist a unique perfect foresight equilibrium converging back to the steady state.*

4. *Assume now, contrary to what was assumed above, that agents internalize that their own consumption choice in period t changes the discount factor, that is, they internalize that θ depends on c_t.*

   Characterize the competitive equilibrium of this economy. Give an intuitive explanation for the differences in equilibrium conditions in the economy with and without internalization.

5. *Characterize the steady state of this economy. Does it exist? Is it unique? Is it the same as in the economy without internalization?*

6. *Characterize the local stability of the steady state. Specifically, suppose that d_{-1} is not equal but close to its steady state value. Under what conditions does there exist a unique perfect foresight equilibrium converging back to the steady state. Express your answer in terms of a condition involving the parameter r and the following four elasticities, $\epsilon_\theta \equiv \frac{\theta'(c)c}{\theta(c)}$, $\epsilon_{\theta\theta} \equiv \frac{\theta''(c)c}{\theta'(c)}$, $\epsilon_c \equiv \frac{U'(c)c}{U(c)}$ and $\epsilon_{cc} \equiv \frac{U''(c)c}{U'(c)}$, evaluated at the steady state value of c_t. Discuss how your result differs from that obtained in question 3 above.*

Exercise 4.3 [Business Cycles in a Small Open Economy with Complete Asset Markets and External Shocks]

Consider the small open economy model with complete asset markets (CAM) studied in this chapter. Suppose that the productivity factor A_t is constant and normalized to 1. Replace the *equilibrium condition* $U_c(c_t, h_t) = \psi_{cam}$ *with the expression*

$$U_c(c_t, h_t) = x_t,$$

where x_t *is an exogenous and stochastic random variable, which can be interpreted as an external shock. Assume that the external shock follows a process of the form*

$$\widehat{x}_t = \rho \widehat{x}_{t-1} + \epsilon_t; \qquad \epsilon_t \sim N(0, \sigma_\epsilon^2),$$

where $\widehat{x}_t \equiv \ln(x_t/x)$ *and* x *denotes the non-stochastic steady-state level of* x_t*. Let* $\rho = 0.9$ *and* $\sigma_\epsilon = 0.02$*. Calibrate all other parameters of the model following the calibration of the CAM model presented in the main body of this chapter. Finally, set the steady state value of* x_t *in such a way that the steady-state level of consumption equals the level of steady-state consumption in the version of the CAM model studied in the main text.*

1. *Produce a table displaying the unconditional standard deviation, serial correlation, and correlation with output of* $\widehat{y}_t$, $\widehat{c}_t$, $\widehat{i}_t$, $\widehat{h}_t$, *and* tb_t/y_t.

2. *Produce a figure with 5 plots depicting the impulse responses to an external shock (a unit innovation in* ϵ_t*) of* $\widehat{y}_t$, $\widehat{c}_t$, $\widehat{i}_t$, $\widehat{h}_t$, *and* tb_t/y_t.

3. *Now replace the values of* ρ *and* σ_ϵ *given above with values such that the volatility and serial correlation of output implied by the model are the same as those reported for the Canadian economy in table 4.2. Answer questions 4.3.a and 4.3.b using these new parameter values.*

4. *Based on your answer to the previous question, evaluate the ability of external shocks (as defined here) to explain business cycles in Canada.*

Exercise 4.4 (A Small Open Economy with an AR(2) TFP Process) *In this question you are asked to show that the SOE-RBC model can predict consumption to be more volatile than output when the productivity shock follows a second-order autoregressive process displaying a hump-shaped impulse response. The theoretical model to be used is the External Debt-Elastic Interest Rate (EDEIR) model presented in section 4.1.1 of the current chapter. Replace the AR(1) process with the following AR(2) specification:*

$$\ln A_{t+1} = 1.42 \ln A_t - 0.43 \ln A_{t-1} + \epsilon_{t+1},$$

where ϵ_t is an i.i.d. random variable with mean zero and standard deviation $\sigma_\epsilon > 0$. Scale σ_ϵ to ensure that the predicted standard deviation of output is 3.08, the value predicted by the AR(1) version of this model. Otherwise use the same calibration and functional forms as presented in the chapter. Download the matlab files for the EDEIR model from `http://www.columbia.edu/~mu2166/closing.htm` *. Then modify them to accommodate the present specification.*

1. *Produce a table displaying the unconditional standard deviation, serial correlation, and correlation with output of output, consumption, investment, hours, the trade-balance-to-output ratio, and the current-account-to-output ratio.*

2. *Produce a 3×2 figure displaying the impulse responses of output, consumption, investment, hours, the trade-balance-to-output ratio, and TFP to a unit innovation in TFP.*

3. *Compare and contrast the predictions of the model under the AR(1) and the AR(2) TFP processes. Provide intuition.*

Exercise 4.5 (Durable Consumption I) *Consider a SOE model with non-durable and durable consumption goods. Let $c_{N,t}$ denote consumption of non-durables in period t and $c_{D,t}$ purchases of durables in period t. The stock of durable consumer goods, denoted s_t, is assumed to evolve over* *time as* $s_t = (1-\delta)s_{t-1} + c_{D,t}$*, where* $\delta \in (0,1]$ *denotes the depreciation rate of durable goods. Households have preferences over consumption,* c_t*, of the form* $\sum_{t=0}^{\infty} \beta^t U(c_t)$*, where* U *is increasing in consumption and concave. Consumption,* c_t*, is a composite of nondurable consumption and the service flow provided by the stock of consumer durables. Specifically, assume that*

$$c_t = \left[(1-\alpha)^{\frac{1}{\eta}} c_{N,t}^{1-\frac{1}{\eta}} + \alpha^{\frac{1}{\eta}} s_t^{1-\frac{1}{\eta}}\right]^{\frac{1}{1-\frac{1}{\eta}}},$$

$\eta > 0$*, and* $\alpha \in (0,1)$*. Households have access to an internationally traded risk-free one-period bond, which pays the interest rate* r_t *when held between periods* t *and* $t+1$*. The relative price of durables in terms of nondurables is one. The household is subject to a borrowing limit that prevents it from engaging in Ponzi schemes. Output, denoted* y_t*, is produced with capital according to a production function of the form* $y_t = F(k_t)$*, where* k_t *denotes physical capital. The capital stock evolves over time as* $k_{t+1} = (1-\delta_k)k_t + i_t$*, where* i_t *denotes investment in period* t *and* δ_k *is the depreciation rate on physical capital.*

1. *Describe the household's budget set.*

2. *State the optimization problem of the household.*

3. *Present the complete set of equilibrium conditions.*

4. *The interest rate is constant over time and equal to* $r_t = r = \beta^{-1} - 1$*. Assume that up to period* -1 *in the economy was in a steady state equilibrium in which all variables were constant and* $d = \bar{d} > 0$*, where* d *denotes net external debt in the steady state.*

   Find the share of expenditures on durables in total consumption expenditures in the steady state in terms of the parameters δ*,* r*,* α*, and* η*. Suggest a strategy for calibrating those four parameters.*

5. *Assume that in period* 0 *the economy unexpectedly receives a positive income shock as a consequence of the rest of the world forgiving part of the country's net foreign debt. Assume that the positive income shock results in a one percent increase in the consumption of nondurables in period 0. Find the percent increase in purchases of durables and in total consumption expenditures in period 0. Compare your answer to the one you would have obtained if all consumption goods were nondurable.*

6. *Continuing to assume that consumption of nondurables increased by one percent, find the change in the trade balance in period 0 expressed as a share of steady state consumption expenditures. Is the response of the trade balance countercyclical? Compare your findings to those you would have obtained if all consumption goods were nondurable. How much amplification is there due to the presence of durables.*

Exercise 4.6 (Durable Consumption II) *Consider an economy populated by a large number of identical households with preferences described by the lifetime utility function*

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{\left[\left(c_t^n - \frac{h_t^\omega}{\omega}\right) s_t^\gamma\right]^{1-\sigma} - 1}{1-\sigma},$$

where c_t^n *denotes consumption of nondurable goods,* h_t *denotes hours worked, and* s_t *denotes the stock of durable consumption goods. The parameter* $\beta \in (0,1)$ *denotes the subjective discount factor,* $\gamma, (\omega - 1), (\sigma - 1) > 0$ *are preference parameters, and* E_t *denotes the expectations operator conditional on information available in period* t.

The law of motion of the stock of durables is assumed to be of the form

$$s_t = (1-\delta)s_{t-1} + c_t^d,$$

where c_t^d denotes durable consumption in period t, and $\delta \in (0,1)$ denotes the depreciation rate. The sequential budget constraint of the household is given by

$$d_t = (1 + r_{t-1})d_{t-1} + c_t^n + c_t^d + \frac{\phi^d}{2}(s_t - s_{t-1})^2 + i_t + \frac{\phi^k}{2}(k_{t+1} - k_t)^2 - A_t k_t^\alpha h_t^{1-\alpha},$$

where d_t denotes debt acquired in period t and maturing in period $t+1$, r_t denotes the interest rate on assets held between periods t and $t+1$, i_t denotes gross investment, k_t denotes the stock of physical capital, and A_t represents a technology factor assumed to be exogenous and stochastic. The parameters $\phi^d, \phi^k > 0$ govern the degree of adjustment costs in the accumulation of durable consumption goods and physical capital, respectively. The parameter α resides in the interval $(0,1)$. The capital stock evolves over time according to the law of motion

$$k_{t+1} = (1-\delta)k_t + i_t.$$

Note that we assume that physical capital, k_t, is predetermined in period t and that investment, i_t, takes one period to become productive capital. By contrast, the stock of consumer durables, s_t is non-predetermined in period t, and expenditures in consumer durables in period t, c_t^d, become productive immediately. Finally, assume that the interest rate is debt elastic,

$$r_t = r^* + \psi \left[e^{\widetilde{d}_t - \bar{d}} - 1 \right],$$

where $\widetilde{d}_t$ denotes the cross-sectional average level of debt per capita, and r^*, $\bar{d}$, and ψ are parameters. The productivity factor A_t evolves according to the expression

$$\ln A_{t+1} = \rho \ln A_t + \epsilon_{t+1},$$

where ϵ_t is a white noise with mean zero and variance σ_ϵ^2, and $\rho \in (0,1)$ is a parameter. Assume that $\beta(1+r^*) = 1$.

1. *Derive the complete set of equilibrium conditions.*

2. *Derive the deterministic steady state. Specifically, find analytical expressions for the steady state values of c_t^n, h_t, s_t, k_{t+1}, d_t, r_t, i_t, tb_t, and ca_t in terms of the structural parameters of the model σ, β, δ, ω, α, γ, r^*, and $\bar{d}$. Here, tb_t and ca_t denote, respectively, the trade balance and the current account.*

3. *Assume the following parameter values: $\sigma = 2$, $\delta = 0.1$, $r^* = 0.04$, $\alpha = 0.3$, and $\omega = 1.455$. Calibrate $\bar{d}$ and γ so that in the steady state the debt to output ratio is 25 percent and the nondurable consumption to output ratio is 68 percent. Report the implied numerical values of γ and $\bar{d}$. Also report the numerical steady state values of r_t, d_t, h_t, k_t, c_t^n, s_t, c_t^d, i_t, tb_t, ca_t, and $y_t \equiv A_t k_t^\alpha h_t^{1-\alpha}$.*

4. *Approximate the equilibrium dynamics using a first-order perturbation technique. In performing this approximation, express all variables in logs, except for the stock of debt, the interest rate, the trade balance, the current account, the trade-balance-to-output ratio, and the current-account-to-output ratio. You are asked to complete the calibration of the model by setting values for ψ, ϕ^d, ϕ^k, ρ, and σ_ϵ to target key empirical regularities of medium-size emerging countries documented in chapter 1 of Uribe's Open Economy Macroeconomics textbook. Specifically, the targets are a standard deviation of output, σ_y, of 8.99 percent, a relative standard deviation of consumption, σ_c/σ_y, of 0.93, a relative standard deviation of gross investment, σ_i/σ_y, of 2.86, a serial correlation of output of 0.84, and a correlation between the trade-balance-to-output ratio and output of -0.24. In general, you will not be able to hit these targets exactly. Instead, you are required to define a distance between the targets and*

their corresponding theoretical counterparts and devise a numerical algorithm to minimize it. Define the distance as follows. Let $z(\psi, \phi^d, \phi^k, \rho, \sigma_\epsilon) \equiv x(\psi, \phi^d, \phi^k, \rho, \sigma_\epsilon) - x^*$*, where* x^* *is the 5x1 vector of empirical targets (the 5 numbers given above) and* $x(\psi, \phi^d, \phi^k, \rho, \sigma_\epsilon)$ *is the 5x1 vector of theoretical counterparts as a function of the parameters. Let* $D(\psi, \phi^d, \phi^k, \rho, \sigma_\epsilon) \equiv \sqrt{z(\psi, \phi^d, \phi^k, \rho, \sigma_\epsilon)' z(\psi, \phi^d, \phi^k, \rho, \sigma_\epsilon)}$ *be the distance between the target and its theoretical counterpart. Report (a) the values of* ψ*,* ϕ^d*,* ϕ^k*,* ρ*, and* σ_ϵ *that you find and (b) complete the following table:*

	Data	*Prediction of the Model*
σ_y	*8.99*	
σ_c/σ_y	*0.93*	
σ_i/σ_y	*2.86*	
corr(y_t, y_{t-1})	*0.84*	
corr$(tb_t/y_t, y_t)$	*-0.24*	

5. *Produce a table displaying the model predictions. The table should contain the unconditional standard deviation, correlation with output, and the first-order serial correlation of output, consumption, investment, consumption of durables, consumption of nondurables, the trade-balance-to-output ratio, and the current-account-to-output ratio. For consumption, consumption of durables, consumption of nondurables, and investment report the standard deviation relative to output. Discuss how well the model is able to explain actual observed second moments that were not targeted in the calibration. Use the second moments reported in table 1.2 of Uribe's textbook to compare the model's predictions to actual data.*

Exercise 4.7 (Complete Markets and The Countercyclicality of the Trade Balance) *Consider a small open economy with access to a complete array of internationally traded state contingent claims. There is a single good, which is freely traded internationally. Let* $r_{t,t+1}$ *denote the period*

t price of a contingent claim that pays one good in a particular state of the world in period $t+1$ divided by the probability of occurrence of that state. The small open economy takes the process for $r_{t,t+1}$ as exogenously given.

Households have preferences over consumption, c_t, and hours, h_t, given by

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{\left(c_t - \frac{h_t^{\omega}}{\omega}\right)^{1-\sigma} - 1}{1-\sigma} \right]; \quad \sigma, \omega > 1,$$

where E_0 denotes the expectations operator conditional on information available in period 0. Households produce goods according to the following production technology

$$A_t k_t^{\alpha} h_t^{1-\alpha},$$

where A_t denotes an exogenous productivity factor, k_t denotes the capital stock in period t, and the parameter $\alpha \in (0,1)$ denotes the elasticity of the production function with respect to capital. Domestic households are the owners of physical capital. The evolution of capital is given by

$$k_{t+1} = (1-\delta)k_t + i_t,$$

where i_t denotes investment in physical capital in period t and $\delta \in (0,1)$ denotes the depreciation rate. In period 0, households are endowed with k_0 units of capital and hold contingent claims (acquired in period -1) that pay d_0 goods in period 0.

1. *State the household's period-by-period budget constraint.*

2. *Specify a borrowing limit that prevents household's from engaging in Ponzi schemes.*

3. *State the household's utility maximization problem. Indicate which variables/processes the*

household chooses and which variables/processes it takes as given.

4. *Derive the complete set of competitive equilibrium conditions.*

5. *Let* $\widehat{x}_t \equiv \ln x_t/x$ *denote the percent deviation of a variable from its non-stochastic steady state value. Assume that in the non-stochastic steady state* $r_{0,t} = \beta^t$ *and* $A_t = 1$*. Show that in response to a positive innovation in technology in period* t*,* $\widehat{A}_t > 0$*, the trade balance will respond countercyclically only if the response in investment in period* t *is positive. Then find the minimum percent increase in investment in period* t *required for the trade balance to decline in period* t *in response to the technology shock. To answer this question use a first-order accurate approximation to the solution of the model. Show that your answer is independent of the expected future value of* A_{t+1}*.*

6. *Compare and contrast your findings in the previous item to the ones derived in chapter 3 for a model with capital accumulation, no depreciation, no capital adjustment costs, inelastic labor supply, and incomplete markets. In particular, discuss how in that model the sign of the impulse response of the trade balance to a positive innovation in the technology shock,* $\widehat{A}_t > 0$*, depended on the persistence of the technology shock. Give an intuitive explanation for the similarities/differences that you identify.*

7. *Now find the size of* $E_t\widehat{A}_{t+1}$ *relative to the size of* $\widehat{A}_t$ *that guarantees that the trade balance deteriorates in period* t *in response to a positive innovation in* A_t *in period* t*. Your answer should be a condition of the form* $\widehat{A}_t < ME_t\widehat{A}_{t+1}$*, where* M *is a function of the structural parameters of the model. In particular, it is a function of* α*,* β*,* δ*, and* ω*. Find the value of* M *for* $\alpha = 1/3$*,* $\delta = 0.08$*,* $\beta^{-1} = 1.02$*, and* $\omega = 1.5$*.*

8. *Discuss to which extend your findings support or contradict Principle I, derived in chapter 3, which states that: "The more persistent are productivity shocks, the more likely is the trade*

balance to experience an initial deterioration in response to a positive technology shock."

9. *How would your answers to questions 5 and 7 change if the period utility function was separable in consumption,* c_t*, and hours,* h_t*?*

Exercise 4.8 [Calibrating the EDEIR Model Using Canadian Data Over the Period 1960-2011] *In section 4.5, we calibrated the EDEIR model using second moments computed using Canadian data over the period 1946-1985. The middle panel of table 4.2 updates the empirical second moments to the period 1960 to 2011. The present exercise uses these empirical regularities to calibrate and evaluate the SOE-RBC model.*

1. *Calibrate the EDEIR model as follows: Set* $\beta = 1/1.04$*,* $\sigma = 2$*,* $\omega = 1.455$*,* $\alpha = 0.32$*,* $\delta = 0.10$*, and* $\overline{d} = 0.7442$*. Set the remaining four parameters,* ρ*,* η*,* ϕ*, and* ψ_1 *to match the observed standard deviations and serial correlations of output and the standard deviations of investment and the trade-balance-to-output ratio in Canada over the period 1960-2011. Approximate the equilibrium dynamics up to first order and use a distance minimization procedure similar to the one used in exercise 4.6. Compare the resulting values for* ρ*,* η*,* ϕ*, and* ψ_1 *with those reported in table 4.1.*

2. *Compute theoretical second moments and present your findings as in the third panel of table 4.2.*

3. *Comment on the ability of the model to explain observed business cycles in Canada over the period 1960-2011.*

4. *Compute the unconditional standard deviation of the productivity shock,* $\ln A_t$ *under the present calibration. Compare this number to the one corresponding to the 1946-1985 calibration presented in section 4.5. Now do the same with the standard deviation of output. Discuss and interpret your findings.*

Exercise 4.9 (A Model of the U.S.-Canada Business Cycle) *Consider a world with two economies, Canada and the United States, indexed by $i = Can, US$, respectively. Suppose that both economies are populated by a large number of identical households with preferences given by*

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{\left[c_t^i - \frac{(h_t^i)^\omega}{\omega}\right]^{1-\sigma} - 1}{1-\sigma},$$

where c_t^i and h_t^i denote, respectively, consumption and hours worked in country i in period t. In both countries, households operate a technology that produces output, denoted y_t^i, using labor and capital, denoted k_t^i. The production technology is Cobb-Douglas and given by

$$y_t^i = A_t^i (k_t^i)^\alpha (h_t^i)^{1-\alpha},$$

where A_t^i denotes a productivity shock in country i, which evolves according to the following AR(1) process:

$$\ln A_{t+1}^i = \rho^i \ln A_t^i + \eta^i \epsilon_{t+1}^i,$$

where ϵ_t^i is an i.i.d. innovation with mean zero and variance equal to one, and ρ^i and η^i are country-specific parameters. Both countries produce the same good. The evolution of capital obeys the following law of motion:

$$k_{t+1}^i = k_t^i + \frac{1}{\phi^i}\left[\left(\frac{i_t^i}{\delta k_t^i}\right)^{\phi^i} - 1\right]\delta k_t^i,$$

where i_t^i denotes investment in country i, and ϕ^i is a country-specific parameter.

Assume that asset markets are complete and that there exists free mobility of goods and financial assets between the United States and Canada, but that labor and installed capital are immobile across countries. Finally, assume that Canada has measure zero relative to the United States, so that the *latter can be modeled as a closed economy.*

Consider the business cycle regularities for Canada for the period 1960 to 2011 shown in exercise 4.8. The following table displays observed standard deviations, serial correlations, and correlations with output for the United States over the period 1960-2011. The source is World Development Indicators. The data are annual and in per capita terms. The series y, c, and i are in logs, and the series tb/y is in levels. All series were quadratically detrended. Standard deviations are measured in percentage points.

Variable	U.S. Data 1960-2011		
	σ_{x_t}	$\rho_{x_t,x_{t-1}}$	ρ_{x_t,GDP_t}
y	2.94	0.75	1.00
c	3.00	0.82	0.90
i	10.36	0.67	0.80
tb/y	0.94	0.79	-0.51

1. *Calibrate the model as follows: Assume that the deterministic steady-state levels of consumption per capita are the same in Canada and the United States. Set* $\beta = 1/1.04$, $\sigma = 2$, $\omega = 1.455$, $\alpha = 0.32$, *and* $\delta = 0.10$. *Set the remaining six parameters,* ρ^i, η^i, *and* ϕ^i, *for* $i = Can, US$, *to match the observed standard deviations and serial correlations of output and the standard deviations of investment in Canada and the United States. Use a distance minimization procedure as in exercise 4.6.*

2. *Approximate the equilibrium dynamics up to first order. Produce the theoretical counterparts of the two tables showing Canadian and U.S. business-cycle regularities.*

3. *Comment on the ability of the model to explain observed business cycles in Canada and the United States.*

4. *Plot the response of Canadian output, consumption, investment, hours, and the trade-balance-to-output ratio to a unit innovation in the Canadian productivity shock. On the same plot,*

show the response of the Canadian variables to a unit innovation to the U.S. productivity shock. Discuss the differences in the responses to a domestic and a foreign technology shock and provide intuition.

5. *Compare, by means of a graph and a discussion, the predicted responses of Canada and the United States to a unit innovation in the U.S. productivity shock. The graph should include the same variables as the one for the previous item.*

6. *Compute the fraction of the volatilities of Canadian output and the trade-balance-to-output ratio explained by the U.S. productivity shock according to the present model. To this end, set* $\eta^{Can} = 0$ *and compute the two standard deviations of interest. Then, take the ratio of these standard deviations to their respective counterparts when both shocks are active.*

7. *This question aims to quantify the importance of common shocks as drivers of the U.S.-Canada business cycle. Replace the process for the Canadian productivity shock with the following one*

$$\ln A_{t+1}^{Can} = \rho^{Can} \ln A_t^{Can} + \eta^{Can} \epsilon_{t+1}^{Can} + \nu \epsilon_{t+1}^{US}.$$

All other aspects of the model are as before. Recalibrate the model using an augmented version of the strategy described above that includes an additional parameter, ν*, and an additional target, the cross-country correlation of output, which in the sample used here is 0.64. Report the new set of calibrated parameters. Compute the variance of Canadian output. Now set* $\nu = 0$ *keeping all other parameter values unchanged, and recalculate the variance of Canadian output. Explain.*

Exercise 4.10 (A EDEIR SOE with GHH Preferences and No Capital) *Consider a small* *open economy populated by an infinite number of identical households with preferences of the form*

$$(1-\sigma)^{-1}\sum_{t=0}^{\infty}\beta^t\left(c_t-\frac{h_t^{\omega}}{\omega}\right)^{1-\sigma},$$

where c_t denotes consumption of a perishable good in period t, h_t denotes labor effort in period t, and $\beta \in (0,1)$, $\sigma > 1$, and $\omega > 1$ are parameters. Each household operates a technology that produces consumption goods according to the relationship

$$y_t = h_t^{\alpha},$$

where y_t denotes output, and $\alpha \in (0,1)$ is a parameter. The household can borrow or lend in international financial markets at the interest rate $r_t = r^* + \rho(\tilde{d}_t)$, where r^* denotes the world interest rate and satisfies $\beta(1+r^*) = 1$. The function $\rho(\tilde{d}_t)$ is a country interest-rate premium in period t, satisfying $\rho(0) = 0$, and $\rho(x) \neq 0$ for $x \neq 0$, where $\tilde{d}_t$ denotes the cross-sectional average debt holdings in period t and is taken as given by the individual household. Let d_t denote the household's debt holdings in period t maturing in $t+1$. Households cannot play Ponzi games.

1. *Write down the household's optimization problem.*

2. *Derive the first-order conditions associated with the household's optimization problem.*

3. *Display the complete set of equilibrium conditions.*

4. *Derive the steady state of the economy. In particular, compute the steady-state values of consumption, hours, output, the trade balance, the current account, and external debt, denoted, respectively, c, h, y, tb, ca, and d.*

5. *Derive analytically a first-order linear approximation of the equilibrium conditions. Express it as a first-order difference equation in the vector $[\hat{d}_{t-1}\ \hat{c}_t]'$, where $\hat{d}_{t-1} \equiv d_{t-1} - d$ and*

$\hat{c}_t \equiv \ln(c_t/c)$.

6. *Derive conditions under which the perfect-foresight equilibrium is locally unique.*

Exercise 4.11 (An SOE-RBC Model with Cobb-Douglas Preferences) *Modify the period utility function of the EDEIR SOE-RBC model of section 4.1 as follows*

$$U(c,h) = \frac{\left[c^{1-\omega}(1-h)^{\omega}\right]^{1-\sigma} - 1}{1-\sigma}.$$

All other features of the model are unchanged.

1. *Derive analytically the steady state of the model.*

2. *Set all parameters of the model as in table 4.1, except for ω. Calibrate ω to ensure that in the deterministic steady state hours equal 1/3 (i.e., to ensure that in the steady state, households spend one third of their time working). Calculate the implied value of ω.*

3. *Produce a table of predicted second moments similar to table 4.2. In performing this step, you might find it convenient to use as a starting point the matlab programs for the EDEIR SOE-RBC model posted online.*

4. *Compare the predictions of the present model with those of its GHH-preference counterpart.*